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# On the Doppler distortion of the sea-wave spectra

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## Abstract

Discussions on a form of a frequency spectrum of wind-driven sea waves just above the spectral maximum have continued for the last three decades. In 1958 Phillips made a conjecture that wave breaking is the main mechanism responsible for the spectrum formation [O.M. Phillips, *J. Fluid Mech.* 4 (1958) 426]. That leads to the spectrum decay  $\sim \omega^{-5}$ , where  $\omega$  is the frequency of waves. There is a contradiction between the numerous experimental data and this spectrum. Experiments frequently show decay  $\sim \omega^{-4}$  [Y. Toba, *J. Oceanogr. Soc. Japan* 29 (1973) 209; M.A. Donelan, J. Hamilton, W.H. Hui, *Phil. Trans. R. Soc. London A315* (1985) 509; P.A. Hwang, et al., *J. Phys. Oceanogr.* 30 (1999) 2753]. There are several ways of the explanation of this phenomenon. One of them (proposed by Banner [M.L. Banner, *J. Phys. Oceanogr.* 20 (1990) 966]) takes into account the Doppler effect due to surface circular currents generated by underlying waves in the Phillips model.

In this article the influence of the Doppler effect on an arbitrary averaged spectrum is considered using both analytic and numerical approaches. Although we mostly concentrated on the very important case of Phillips model, the developed technique and general formula can be used for the analysis of other spectra.

For the particular case of Phillips spectra we got analytic asymptotics in the vicinity of spectral maximum and for high frequencies. Results were obtained for two most important angular dependences of the spectra: isotropic and strongly anisotropic. Together with the analytic investigation we performed numerical calculations in a wide range of frequencies. Both high and low frequency asymptotics are in very good agreement with the numerical results.

It was shown that at least at low frequencies, the correction to the spectrum due to the Doppler shift is negligible. At high frequencies there is an asymptotic with tail  $\sim \omega^{-3}$ .

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## 1. Introduction

There is a long discussion in the scientific world about the form of wind generated sea wave spectrum in the “universal” region of frequencies  $\omega \gtrsim \omega_p$ , where  $\omega_p$  is the frequency of the spectral maximum.

There are at least two main points of view. The first one proposed by O. Phillips in 1958 [1] gives us the Phillips’ spectrum in universal region

$$F(\omega) \simeq \frac{\alpha g^2}{\omega^5}, \quad (1)$$

where  $\alpha$  is a dimensionless constant, which according to experimental data appears to be rather small  $\alpha \simeq 0.01$ . The second point of view introduced by V.E. Zakharov and N.N. Filonenko [8] gives us in universal region following spectrum

$$F(\omega) \simeq \frac{\beta \varepsilon^{\frac{1}{2}} g v}{\omega^4}, \quad (2)$$

where  $\beta$  — is a dimensionless constant,  $v$  — is wind velocity,  $\varepsilon \simeq \rho_a/\rho_w$  — is the ratio of ocean water and atmosphere densities.

The formulae (1) and (2) are based upon completely opposite propositions. The Phillips’ spectrum (1) takes place if the spectrum at the high frequency region is mainly determined by

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the appearance of sharp crests and wave breaking, which are strongly nonlinear phenomena.

On the other side, spectrum (2) is based on presumption of small mean steepness —  $\mu \ll 1$  and based on the weakly nonlinear waves interaction. Usually, for mature sea  $\mu \simeq 0.1$ .

Following this ideology the wind generated waves' ensemble is described by Hasselmann kinetic equation [6,7]

$$\frac{\partial N}{\partial t} = S_{nl} + p^+ + p^-. \quad (3)$$

Zakharov–Filonenko spectrum appears as an exact solution of the particular case of Eq. (3)

$$S_{nl} = 0, \quad (4)$$

this is a classical case of Kolmogorov–Zakharov (KZ) spectra. The theory of weak turbulence is far advanced both analytically and numerically [8–18].

In the region of moderate frequencies in field experiments with mature sea the  $\omega^{-4}$  spectrum is dominating with confidence [2–4]. From this fact the question of physical interpretation of spectrum  $F(\omega) \simeq \omega^{-4}$  becomes very important. Following Zakharov and Filonenko this spectrum is just weakly turbulent KZ-spectrum. There are other explanations, however.

One of them was proposed by Phillips in 1986 [19]. He made the supposition that the spectrum  $\omega^{-4}$  is a result of a balance of all three terms of Eq. (3)

$$S_{nl} \simeq p^+ + p^-. \quad (5)$$

This hypothesis is unlikely to be treated as a theory, because the expression of  $p^-$  is almost completely unknown. There are only very rough empirical formulas. However, in that paper this expression was taken in the form that gives us the balance in Eq. (5). For now there are no strong arguments in favor of this conjecture.

Another explanation was proposed by Banner [5] and developed by Donelan [20]. They considered the  $\omega^{-4}$  spectrum as “an artifact of the observation of time histories at a point brought about by Doppler shifting of the short waves riding on the orbital currents of the long waves” — citation from [20].

In the paper [5] yet another explanation was proposed. It is based on taking into account the angle dependence of spatial spectra. Today there is no common opinion about the exact form of such a distribution. For example angular distributions obtained by Donelan et al. [3] and Hwang et al. [4] in field experiments differ significantly. Also, the universality of angular dependence is not obvious yet.

The main goal of this paper is to find a frequency spectra of waves from known space-spectra in the presence of the Doppler effect. Let us suppose that spatial spectrum of the short waves  $\Phi(k)$  is known. The problem is to derive the frequency spectrum  $F(\omega)$  using only  $\Phi(k)$ . Normally we have to assume that waves obey the same dispersion relation as in the linear case. Thus the following simple expression takes place for positive frequency part of the energy spectrum

$$F(\omega) = \int \Phi(k)\delta(\omega - \omega(k))dk.$$

However in the presence of long waves the dispersion relation of the short waves modifies due to Doppler effect at the orbital velocity's field  $\vec{v}$  of the long wave component

$$\omega(k) \rightarrow \omega(k) + (\vec{k}\vec{v}). \quad (6)$$

In the well-known article by Kitaigorodskii, Krasitskii, Zaslavskii [21] it was considered that  $\vec{v}$  is a random function on time. In fact, it is a quasi-periodic function with random envelope. In the present paper we simplify the problem and assume that  $\vec{v}$  is a pure periodic function. Another word, we study modification of the spectrum due to a presence of a long monochromatic wave. This assumption gives us an opportunity to develop the analytical theory up to explicit formulae.

The present paper is devoted to the quantitative theory of this, without any doubt, very important phenomenon. We consider it in a very simplified way, supposing that  $\vec{v}$  is a sinusoidal function of time, i.e. assuming that the long wave is strictly monochromatic. In spite of simplicity this assumption no doubt gives us the correct order of magnitude.

In Section 2 we introduced the common notations (following Phillips). In Section 3 the general formula for spectrum (taking into account Doppler shifting) is derived. It is convenient to introduce a new set of dimensionless variables, this is considered in Section 4. In Section 5 we develop an important case of dispersion  $\omega^2 = gk$  (“deep water” surface gravity waves) where  $g$  is the gravity acceleration,  $k$  is the wave number equal to  $2\pi/\lambda$ . Here we also apply developed technique to the important case of Phillips model. Section 6 is devoted to numerical results. The conclusion is placed in Section 7.

In the case of relatively small background velocities (with respect to phase velocity) the Doppler effect appears to be too weak (the first correction to the Phillips spectrum is proportional to  $\sim (v\omega/g)^2$ ) in a wide range of frequencies. In an opposite case of high speed background Doppler shifting is very important and gives us at high frequencies another asymptotic tail  $\sim \omega^{-3}$ . Numerical results are in good agreement with analytic asymptotics and give us information about spectra distortion in the intermediate range of frequencies.

## 2. Formulation of the problem and notations

Following the notations of Phillips, the wave spectrum for homogeneous stationary wave field can be introduced as follows

$$X(\vec{k}, \omega) = (2\pi)^{-3} \int \int_{-\infty}^{+\infty} \rho(\vec{r}, t) \exp[-i(\vec{k}\vec{r} - \omega t)] d\vec{r} dt, \quad (7)$$

where  $\rho(\vec{r}, t) = \overline{\xi(\vec{x}, t_0)\xi(\vec{x} + \vec{r}, t_0 + t)}$  is the covariance of the surface displacement  $\xi(\vec{x}, t)$ ,  $\vec{r}$  is the spatial separation vector,  $t$  is the time separation,  $\vec{k} = (k_1, k_2) = (k, \theta)$  is the wavenumber vector (the second expression is the wavevector representation in polar coordinates) and  $\omega$  is the radian frequency. Obviously it can be rewritten in terms of Fourier transforms of corresponding functions

$$\left\langle \xi(\vec{k}, \omega) \xi^*(\vec{k}', \omega') \right\rangle = X(\vec{k}, \omega) \delta_{\vec{k}-\vec{k}'} \delta_{\omega-\omega'}, \quad (8)$$

$$X(-\vec{k}, -\omega) = X(\vec{k}, \omega).$$

It is convenient to introduce additional spectra which are reduced forms of Eq. (7)

$$\bar{\Phi}(\vec{k}) = 2 \int_0^{+\infty} X(\vec{k}, \omega) d\omega, \quad (9)$$

$$F(\omega) = 2 \int_0^{2\pi} \int_0^{+\infty} X(\vec{k}, \omega) k dk d\theta. \quad (10)$$

In general case it is impossible to express  $F(\omega)$  in terms of  $\bar{\Phi}(\vec{k})$ .

There are many complex mechanisms of long and short waves interaction. They were considered by Phillips [22] at the vicinity of spectral peak frequency  $\omega_p$ . According to this paper, the first correction terms taking into account variable effective gravity and other nonlinear effects are proportional to steepness (which is supposed to be small in our consideration). In our case, when the Doppler velocity does not depend on steepness, the Doppler effect is also insensitive to roughness of the waves. In the special case, when we consider influence of a periodic current due to spectral peak we discuss this issue in details. In this article we limit our consideration only by Doppler effect. In this sense our model is different with respect to one used by Banner [5]. Nevertheless, the results obtained are in an agreement with experimental data by Donelan [3] which show no or indistinguishable dependence of spectral slope with moderate change of steepness. The more detailed consideration can be an inspiration for future investigations.

Following Appendix A we shall use linearized relations for spectra. For an isotropic spectrum we have (A.16)

$$X_{is}(\vec{k}, \omega) = \frac{1}{2} \bar{\Phi}(\vec{k}) (\delta(\omega - \omega_k) + \delta(\omega + \omega_k)), \quad (11)$$

where  $\omega_k = \omega(k)$  is the dispersion law and  $\delta$  is the Dirac delta-function.

In the weakly nonlinear approximation we consider interaction between positive and negative-frequency parts of the spectra to be negligible. This simplification let us to find dependence between spatial and frequency spectra. From now on we shall consider positive-frequency parts of spectra.

In this case in the isotropic case we shall get

$$X_{is}(\vec{k}, \omega) = \frac{1}{2} \bar{\Phi}(k) \delta(\omega - \omega_k). \quad (12)$$

In a unidirectional (or strongly anisotropic) case, when all waves propagate in one direction, corresponding to angle  $\theta_0$ , the relation is a little bit different (A.17)

$$X_{an}(\vec{k}, \omega) = \bar{\Phi}(k) \delta(\theta - \theta_0) \delta(\omega - \omega_k). \quad (13)$$

From now on we shall, for the sake of universality, hide the multiplier 1/2 in isotropic case in the angular dependence of the spectrum. Finally, we get simplified relation for the spectra

$$X(\vec{k}, \omega) = \bar{\Phi}(\vec{k}) \delta(\omega - \omega_k). \quad (14)$$

Now one can rewrite Eq. (10) in the following form

$$F(\omega) = 2 \int_0^{+\infty} \int_0^{2\pi} \bar{\Phi}(k, \theta) \delta(\omega - \omega_k) d\theta k dk. \quad (15)$$

Using the rules of integration of  $\delta$ -function one can obtain

$$F(\omega) = \frac{2k}{|\omega'(k)|} \int_0^{2\pi} \bar{\Phi}(k, \theta) d\theta, \quad (16)$$

$$k = k(\omega),$$

where  $k(\omega)$  is the inverse function for  $\omega(k)$ .

In the deep water case  $\omega = \sqrt{gk}$

$$k = \frac{\omega^2}{g}, \quad \omega'(k) = \frac{1}{2} \sqrt{\frac{g}{k}} = \frac{1}{2} \frac{g}{\omega}$$

$$F(\omega) = \frac{4\omega^3}{g^2} \int_0^{2\pi} \bar{\Phi}\left(\frac{\omega^2}{g}, \theta\right) d\theta. \quad (17)$$

For an isotropic spectrum ( $\bar{\Phi}(k, \theta) = \bar{\Phi}(k)/2$ )

$$F_{is}(\omega) = \frac{4\pi\omega^3}{g^2} \bar{\Phi}\left(\frac{\omega^2}{g}\right). \quad (18)$$

For a strongly anisotropic spectrum ( $\bar{\Phi}(k, \theta) = \bar{\Phi}(k)\delta(\theta - \theta_0)$ )

$$F_{an}(\omega) = \frac{4\omega^3}{g^2} \bar{\Phi}\left(\frac{\omega^2}{g}\right). \quad (19)$$

If the spectrum is determined by discontinuities of spatial derivative (wedges) caused by wave breaking according to Phillips [1], we have

$$\xi \sim \frac{1}{k^2}, \quad \bar{\Phi}(k) \sim \frac{1}{k^4}. \quad (20)$$

In the isotropic case substitution Eq. (20) to Eq. (18) gives

$$F_{is0}(\omega) = 4\pi\alpha \frac{g^2}{\omega^5}. \quad (21)$$

Similar calculation for Eq. (19) gives us the following result

$$F_{an0}(\omega) = 4\alpha \frac{g^2}{\omega^5}. \quad (22)$$

Here  $\alpha$  is the so-called Phillips' constant.

There is a recent work by Kuznetsov [23] which gives us different spatial spectra for waves wedges due to more accurate analysis. In this work we shall use spectrum (20) as an example, because spectra of this type are still widely used in oceanology. It should be stressed, that general formulae given below are valid for the Doppler distortion of any waves spectra in a weakly nonlinear approximation.

### 3. General formula for spectrum in the presence of Doppler effect

Let us assume that we study waves on the background of periodic current  $\vec{v}(t)$  beneath the surface. Taking into account the Doppler effect in the dispersion equation one can get the following substitution

$$\omega(\vec{k}) \longrightarrow \omega(\vec{k}) + \vec{k}\vec{v}(t).$$

Suppose that this current is described by a one-dimensional periodic function

$$v(t) = v \cos(\omega_0 t).$$

This allows us to rewrite our substitution in the following form

$$\omega(k) \longrightarrow \omega(k) + kv \cos(\omega_0 t) \cos \psi, \quad (23)$$

where  $\psi$  is the angle formed by  $\vec{k}$  and  $\vec{v}$  vectors. Without loss of generality one can take  $\psi = \theta$ .

We can now find a range of velocities when we can consider only positive frequencies. In general such conditions gives us  $v \leq w(k)/k = c_p$ , where  $c_p$  is the phase velocity. In the special case of the Doppler shift due to spectral peak currents, using (A.6) one can get

$$kA_p \omega_p \leq \omega.$$

Which gives us

$$\omega \leq \frac{1}{\mu} \omega_p, \quad (24)$$

where  $\mu = k_p A_p$  is a steepness of spectrum peak wave. For the usual value  $\mu = 0.1$  we get  $\omega \leq 10\omega_p$  which is more then enough for interpretation any open sea experimental data. Even in the case of quite rough sea  $\mu = 0.2$  we cover frequency bandwidth of most of current open water experiments.

Using expression (23) one can write average value of expression (15) as

$$F(\omega) = \frac{2}{T} \int_0^T \int_0^{+\infty} \int_0^{2\pi} \Phi(k, \theta) \delta(\omega - \omega(k) - kv \cos(\omega_0 t) \cos \theta) d\theta k dk dt, \quad (25)$$

where  $T$  is the period equal to  $2\pi/\omega_0$ . It is useful to take the average value of this expression with respect to  $t$  as the first step. Let us denote this factor to the part of the integral as  $M$

$$F(\omega) = \int_0^{+\infty} \int_0^{2\pi} \Phi(k, \theta) M(k, \theta) d\theta k dk,$$

$$M(k, \theta) = \frac{1}{\pi} \int_0^T \delta(\omega - kv \cos \omega_0 t \cos \theta - \omega_k) d(\omega_0 t).$$

The  $\delta$ -function in this expression gives us an equation for  $t_0$

$$\omega - \omega_k = kv \cos \theta \cos(\omega_0 t_0), \quad (26)$$

This equation have the following roots

$$\omega_0 t_0 = \pm \arccos \frac{\omega - \omega_k}{kv \cos \theta}, \quad (27)$$

when the following inequality is satisfied

$$\left| \frac{\omega - \omega_k}{kv \cos \theta} \right| < 1. \quad (28)$$

Using (27) and delta function integration rules one can get the function M

$$M(k, \theta) = \frac{2}{\pi \left| kv \cos \theta \sqrt{1 - \frac{(\omega - \omega_k)^2}{k^2 v^2 \cos^2 \theta}} \right|}$$

$$= \frac{2}{\pi \sqrt{k^2 v^2 \cos^2 \theta - (\omega - \omega_k)^2}}. \quad (29)$$

Using this expression one can get

$$F(\omega) = \frac{2}{\pi} \int \int \frac{\Phi(k, \theta) d\theta k dk}{\sqrt{k^2 v^2 \cos^2 \theta - (\omega - \omega(k))^2}}. \quad (30)$$

The limits of integration have to be chosen in order to satisfy inequality (28).

#### 4. Convenient set of dimensionless variables

For further consideration it is convenient to introduce a new variable and a parameter

$$\zeta = \frac{kv^2}{g}, \quad \lambda = \frac{v\omega}{g}. \quad (31)$$

One can see that  $\zeta$  is a dimensionless analogue of  $k$ , and  $\lambda$  is an analog of  $\omega$ . In these variables the spectrum Eq. (30) takes the following relatively simple form

$$F(\lambda) = \frac{2}{\pi} \frac{g}{v^3} B \int_0^{+\infty} \int_0^{2\pi} \frac{\Phi(\zeta, \theta) d\zeta d\theta}{\sqrt{\cos^2 \theta - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2}}, \quad (32)$$

where  $B$  is the dimensional constant which has an origin in substitution  $k \rightarrow \zeta$  in  $\Phi(k)$ . For instance in the Phillips spectrum case Eq. (20) one can get

$$\Phi(k) = \alpha k^{-4} = \alpha \left(\frac{\zeta g}{v^2}\right)^{-4} = \alpha \frac{v^8}{g^4} \zeta^{-4} = B \zeta^{-4},$$

thus the constant takes the form  $B = \alpha v^8/g^4$ .

The physical meaning of the variable  $\lambda$  becomes more clear if we transform it to the following form

$$\lambda = \frac{\omega}{\omega_p} \frac{v}{c_p}, \quad (33)$$

where  $\omega_p = g/c_p$  is the peak-frequency (this is the only frequency-like scale in the model) and  $c_p$  is the phase velocity corresponding to the spectral peak.

For the case of strongly anisotropic spectrum ( $\Phi(k, \theta) = \Phi(k)\delta(\theta)$ ) one can easily obtain the following result by a simple substitution in Eq. (32)

$$F_{an}(\lambda) = \frac{2}{\pi} \frac{g}{v^3} B \int_0^{+\infty} \frac{\Phi(\zeta) d\zeta}{\sqrt{1 - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2}}. \quad (34)$$

If the spectrum is isotropic ( $\Phi(k, \theta) = \Phi(k)/2$ ) things become a little bit more complex. Following Eq. (32) we have

$$F_{is}(\lambda) = \frac{4}{\pi} \frac{g}{v^3} B \int_0^{+\infty} \Phi(\zeta) d\zeta \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2 \theta - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2}}, \quad (35)$$

An analysis of the  $\theta$ -part gives [24]

$$\begin{aligned} & \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2 \theta - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2}} \\ &= \frac{1}{\sqrt{1 - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2}} K \left( \frac{1}{\sqrt{1 - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2}} \right) \\ &= K \left( \sqrt{1 - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2} \right) \\ &+ \frac{i}{\sqrt{1 - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2}} K \left( \frac{\lambda - \lambda_\zeta}{\zeta} \right), \end{aligned} \quad (36)$$

where  $K$  is the complete elliptic integral of the first kind. Let us have a look at inequality (28). This is nothing but a condition which guarantees that our spectrum is a pure real function. In addition:

$$\left| \frac{\lambda - \lambda_\zeta}{\zeta} \right| \leq \left| \frac{\lambda - \lambda_\zeta}{\zeta \cos \theta} \right| \leq 1.$$

It means that all functions in (36) are real and we can omit the part with imaginary unit  $i$ . After substitution Eq. (36) into Eq. (35) one can obtain

$$F_{is}(\lambda) = \frac{4}{\pi} \frac{g}{v^3} B \int_0^{+\infty} \Phi(\zeta) d\zeta K \left( \sqrt{1 - \left(\frac{\lambda - \lambda_\zeta}{\zeta}\right)^2} \right). \quad (37)$$

Expressions (34) and (37) are valid for an arbitrary dispersion relation  $\lambda_\zeta = \omega(k(\zeta))v/g$ .

### 5. Deep water case

For further study of the spectrum we have to introduce the explicit form of the dispersion relation. Here we consider one of the most important cases, i.e. deep water surface gravity waves. In the case of a deep water  $\omega_k = \sqrt{gk}$ , correspondently  $\lambda_\zeta = \sqrt{\zeta}$ . Let us consider an isotropic case

$$F_{is}(\lambda) = \frac{4}{\pi} \frac{g}{v^3} B \int_0^{+\infty} \Phi(\zeta) d\zeta K \left( \sqrt{1 - \left(\frac{\lambda - \sqrt{\zeta}}{\zeta}\right)^2} \right). \quad (38)$$

Now we have to take into account condition (28)

$$\zeta^2 - (\lambda - \sqrt{\zeta})^2 \geq 0. \quad (39)$$

Solutions of this inequality give us integration domains

$$\begin{aligned} & \text{when } \lambda \leq \frac{1}{4} \\ & \left( -\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda} \right)^2 < \zeta < \left( \frac{1}{2} - \sqrt{\frac{1}{4} - \lambda} \right)^2, \end{aligned} \quad (40)$$

$$\text{when } \lambda > \frac{1}{4}, \quad \left( -\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda} \right)^2 < \zeta < +\infty,$$

so we have following expressions for spectra

$$\begin{aligned} & \text{when } \lambda \leq \frac{1}{4}, \quad F_{is}(\lambda) = \frac{4}{\pi} \frac{g}{v^3} B \left[ \int_{\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda}\right)^2}^{\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}\right)^2} \Phi(\zeta) \right. \\ & \times K \left( \sqrt{1 - \left(\frac{\lambda - \sqrt{\zeta}}{\zeta}\right)^2} \right) d\zeta \\ & \left. + \int_{\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}\right)^2}^{+\infty} \Phi(\zeta) K \left( \sqrt{1 - \left(\frac{\lambda - \sqrt{\zeta}}{\zeta}\right)^2} \right) d\zeta \right]; \end{aligned} \quad (41)$$

$$\begin{aligned} & \text{when } \lambda > \frac{1}{4}, \quad F_{is}(\lambda) = \frac{4}{\pi} \frac{g}{v^3} B \int_{\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda}\right)^2}^{+\infty} \Phi(\zeta) d\zeta \\ & \times K \left( \sqrt{1 - \left(\frac{\lambda - \sqrt{\zeta}}{\zeta}\right)^2} \right). \end{aligned} \quad (42)$$

Let us explore special case, when  $\Phi(\zeta)$  is a quickly going down to zero function, e.g. Phillips spectrum  $\Phi(\zeta) = 1/\zeta^4$ . In this case we can take into account only the first term in Eq. (41) at  $\lambda < 1/4$ . Calculation of the expansion of the spectrum correction is placed in Appendix B and (B.7) gives us the following result

$$F_{is}(\lambda) = F_{is0} \left( 1 + \frac{1}{4} \lambda^2 + \dots \right), \quad (43)$$

where

$$F_{is0} = 4\pi\alpha \frac{v^5}{g^3} \frac{1}{\lambda^5}, \quad (44)$$

this is simply the Phillips spectrum Eq. (21) at the absence of Doppler effect for isotropic distribution.

For the one-directional spectra we can use exactly the same approach, but all calculations become simpler. The domains of integrations (which correspond to Eqs. (41) and (42)) are the following

$$\begin{aligned} & \text{when } \lambda \leq \frac{1}{4}, \\ & F_{an}(\lambda) = \frac{2}{\pi} \frac{g}{v^3} B \left( \int_{\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda}\right)^2}^{\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}\right)^2} \frac{\Phi(\zeta) d\zeta}{\sqrt{1 - \left(\frac{\lambda - \sqrt{\zeta}}{\zeta}\right)^2}} \right. \\ & \left. + \int_{\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}\right)^2}^{+\infty} \frac{\Phi(\zeta) d\zeta}{\sqrt{1 - \left(\frac{\lambda - \sqrt{\zeta}}{\zeta}\right)^2}} \right); \end{aligned} \quad (45)$$

$$\begin{aligned} & \text{when } \lambda > \frac{1}{4}, \\ & F_{an}(\lambda) = \frac{2}{\pi} \frac{g}{v^3} B \int_{\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda}\right)^2}^{+\infty} \frac{\Phi(\zeta) d\zeta}{\sqrt{1 - \left(\frac{\lambda - \sqrt{\zeta}}{\zeta}\right)^2}}. \end{aligned} \quad (46)$$

The detailed analysis of this case is situated in Appendix B and (B.9) gives us the following formula

$$F_{an}(\lambda) = F_{an0} \left( 1 + \frac{1}{2}\lambda^2 + \dots \right), \quad (47)$$

where

$$F_{an0} = 4\alpha \frac{v^5}{g^3} \frac{1}{\lambda^5}, \quad (48)$$

this is just the Phillips spectrum Eq. (22) at the absence of Doppler effect for strongly anisotropic distribution.

Let us try to evaluate the range of applicability of the obtained results. As it was postulated in Eq. (33) we can state

$$\lambda = \frac{\omega}{\omega_p} \frac{v}{c_p} < \frac{1}{4}.$$

Thus, for example, if we take  $\varepsilon = v/c_p = 0.05$  (a reasonable value of wind drift current (Banner, 1990), or if we consider the influence of high frequency wave on the far tail of spectrum) one can obtain the following inequality

$$\omega < 5\omega_p. \quad (49)$$

If we consider a Doppler shift due to long waves corresponding to the vicinity of spectral maximum the result will be different. Following (A.6)  $v = A_p \omega_p$ .

In this case the physical meaning of our dimensionless variables is quite straightforward

$$\zeta = \mu^2 \frac{k}{k_p}, \quad \lambda = \mu \frac{\omega}{\omega_p}, \quad (50)$$

here  $k_p$  and  $\omega_p$  are wavenumber and circular frequency corresponding to spectrum peak, and  $\mu = A_p k_p$ , characteristic steepness of spectrum peak. Usually this value is about  $\mu \simeq 0.1$ . This gives us estimations

$$\omega < 2.5\omega_p. \quad (51)$$

Although this more or less corresponds to the frequency bandwidth of many experiments, we need to find an asymptotic of Phillips frequency spectrum  $F(\lambda)$  at high frequencies

$$\lambda \gg 1/4.$$

This condition simplifies situation significantly. In the isotropic case one can get

$$F_{is}(\lambda) \simeq \frac{4}{\pi} \frac{g}{v^3} B \int_{\lambda}^{+\infty} \Phi(\zeta) d\zeta K \left( \sqrt{1 - \left(\frac{\lambda}{\zeta}\right)^2} \right). \quad (52)$$

In the anisotropic case

$$F_{an}(\lambda) \simeq \frac{2}{\pi} \frac{g}{v^3} B \int_{\lambda}^{+\infty} \frac{\Phi(\zeta) d\zeta}{\sqrt{1 - \left(\frac{\lambda}{\zeta}\right)^2}}. \quad (53)$$

Here we left only the term of the order of  $\lambda$  under square root sign having supposed a fast decay of spectrum  $\Phi(\zeta)$ . An obvious substitution  $x = \lambda/\zeta$  in the case of Phillips spectrum

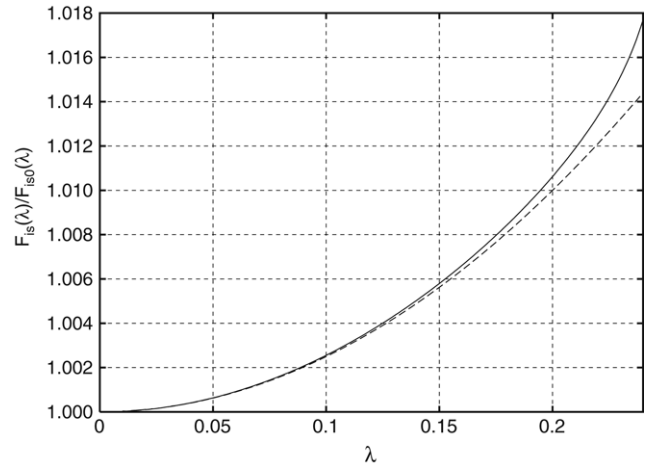


Fig. 1. Calculated spectrum in isotropic case compensated with corresponding Phillips spectrum  $F_{is}(\lambda)/F_{is0}(\lambda)$  (solid line) and analytic estimation (43) in the case of  $\lambda < 1/4$  (dashed line).

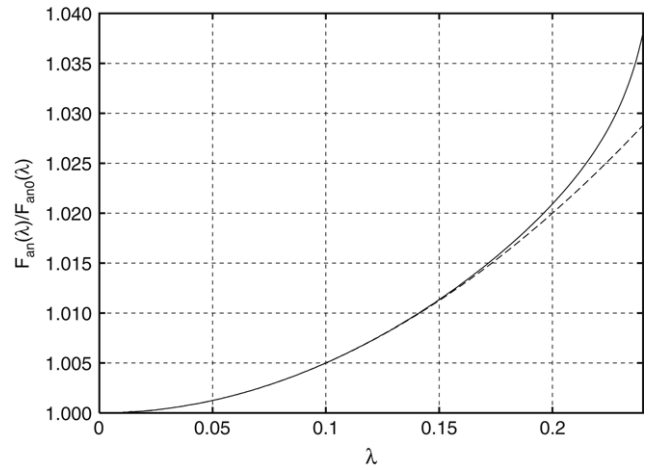


Fig. 2. Calculated spectrum in strongly anisotropic case compensated with corresponding Phillips spectrum  $F_{an}(\lambda)/F_{an0}(\lambda)$  (solid line) and analytic estimation (47) in the case of  $\lambda < 1/4$  (dashed line).

$\Phi(\zeta) = \zeta^{-4}$  immediately leads to answers

$$F_{is}(\lambda) \simeq \frac{4}{\pi} \frac{\alpha v^5}{g^3} \frac{1}{\lambda^3} \int_0^1 x^2 dx K(\sqrt{1-x^2}) = \frac{\alpha v^5}{g^3} \frac{\pi}{4} \frac{1}{\lambda^3}, \quad (54)$$

$$F_{an}(\lambda) \simeq \frac{2}{\pi} \frac{\alpha v^5}{g^3} \frac{1}{\lambda^3} \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{\alpha v^5}{g^3} \frac{1}{2} \frac{1}{\lambda^3}, \quad (55)$$

for isotropic and anisotropic spectra respectively.

As we can see slopes of the spectra changed and one could expect to find in the middle range of frequencies long enough domain of  $F(\omega) \sim \omega^{-4} \sim \lambda^{-4}$  spectrum dependence. To develop this hypothesis we need to perform numerical calculation of integrals (41), (42), (45) and (46) in the whole range of parameters.

## 6. Numerical results

At the first stage let us compare results in the case  $\lambda < 1/4$  with our estimations. Results of these calculations are represented in Figs. 1 and 2 for isotropic and anisotropic cases.

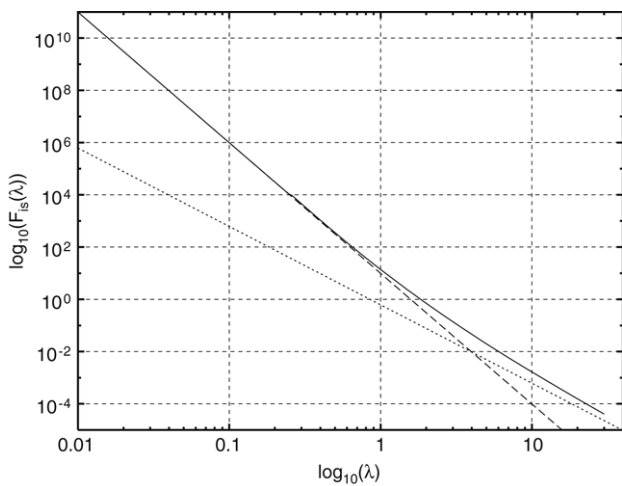


Fig. 3. Calculated spectrum in isotropic case  $F_{is}(\lambda)$  (solid line), Phillips spectrum  $F_{is0}(\lambda)$  (dashed line) and asymptotic (54) corresponding to  $\lambda \gg 1/4$  (dotted line).

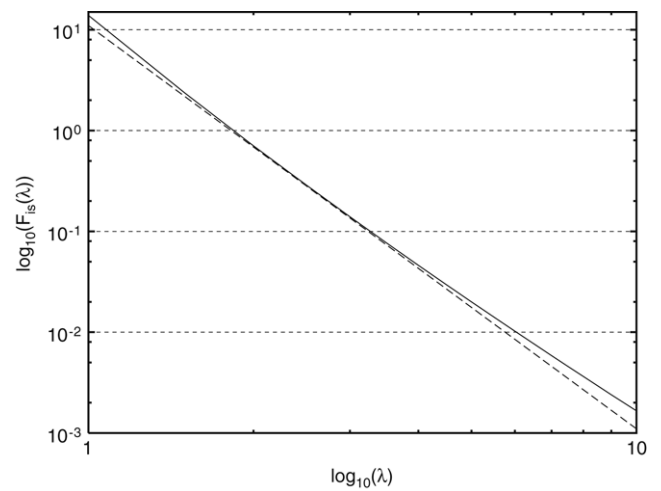


Fig. 5. Calculated spectrum in isotropic case  $F_{is}(\lambda)$  (solid line) and frequently observed spectrum  $\lambda^{-4}$  (dashed line). The spectrum slope is close to  $\omega^{-4}$  for  $2 < \lambda < 3$ .

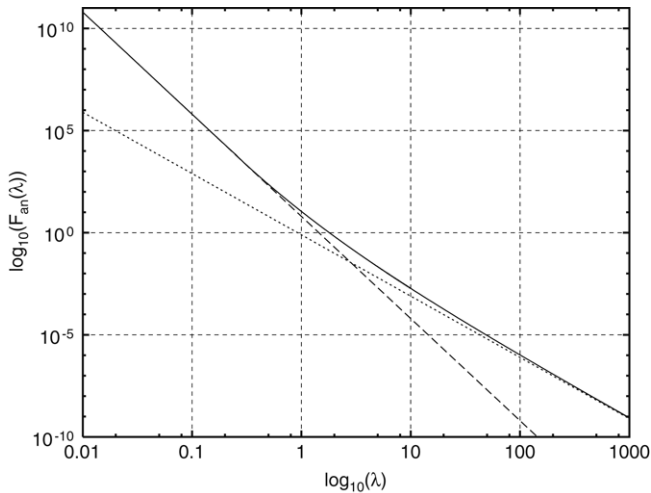


Fig. 4. The calculated spectrum in the strongly anisotropic case  $F_{an}(\lambda)$  (solid line), the Phillips spectrum  $F_{an0}(\lambda)$  (dashed line) and the asymptotic (55) corresponding to  $\lambda \gg 1/4$  (dotted line).

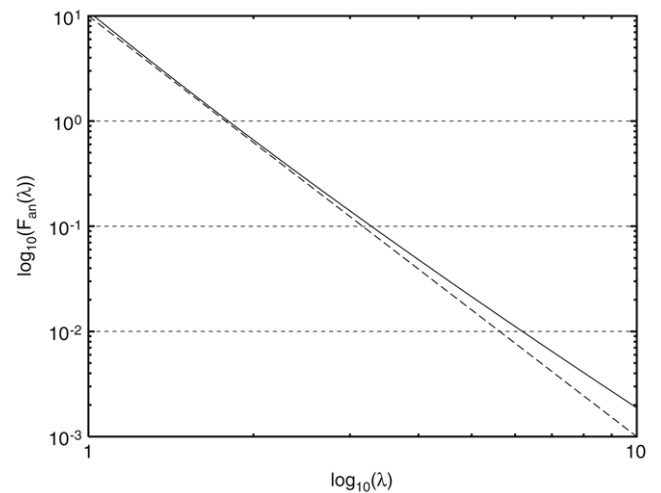


Fig. 6. Calculated spectrum in strongly anisotropic case  $F_{an}(\lambda)$  (solid line) and frequently observed spectrum  $\lambda^{-4}$  (dashed line). The spectrum slope is close to  $\omega^{-4}$  for  $1.5 < \lambda < 2$ .

One can see, that numerical calculation gives us the same result – there is a very small deviation from Phillips spectra. Discrepancies of analytic and numerical results at the edge of the domain ( $\lambda \simeq 1/4$ ) show that we cannot use only several terms expansions and have to take into account further terms in square root decompositions used in Appendix B.

To understand results at the whole range of parameters it is convenient to use double logarithm representation. One can see in Figs. 3 and 4 that Phillips spectrum remains untouched in a wide range of parameters and then transforms to asymptotic (54) and (55) respectively.

Now we can consider the interesting problem of Doppler distortion due to the influence of spectrum peak. In this special case we are interested in the region of parameters  $1 < \lambda < 10$  corresponding to the switch of the spectrum slope. Following our results in such a situation, one could expect formation of  $\omega^{-4} \sim \lambda^{-4}$  spectrum due to transfer from Phillips  $\lambda^{-5}$  to  $\lambda^{-3}$

tail. Let us consider our results in detail. It is clear from Figs. 5 and 6 that there is no such a region. Universal spectra  $\omega^{-4}$  only touch our curve. It means that Doppler distortion due to spectral peak waves influence cannot be considered as a mechanism of transformation of the analytic Phillips spectra  $\omega^{-5}$  to the frequently observed  $\omega^{-4}$ .

From the Figs. 3 and 4 one can notice that we have  $\omega^{-5}$  tail at least up to  $\lambda = 0.5$ . Following physical meaning of  $\lambda$  (33) we have that  $\omega^{-5}$  tail stays the same at least for

$$\omega \leq 5\omega_p,$$

in the case when spectrum peak steepness is about  $\mu \simeq 0.1$ . In fact we observe tail close to  $\omega^{-4}$  only in the vicinity of the points  $\lambda \simeq 2$ , which corresponds at such value of steepness to  $\omega = 20\omega_p$  – beyond the frequency bandwidth of even the current state of the art water tank experiments [25].



We would like to stress, that change of spectrum peak steepness does not affect general results of the paper, but shifts the frequency region where we observe turn from  $\omega^{-5}$  to  $\omega^{-3}$  tail.

In spite of the simplicity of formulae under consideration, they are not suitable for reliable numerical calculations (especially in the case  $\lambda \gg 1/4$ ) using common numerical packages, because of points on both sides of integration interval and inside of it which give singularities of expression under integral sign, causing very slow convergence of algorithm. Nevertheless all these obstacles can be avoided or eliminated using standard numerical analysis techniques. As an examples we represented direct formula for integration in the  $\lambda > 1/4$  case of strongly anisotropic spectrum in [Appendix C](#)

### 7. Conclusion

In this paper we considered the influence of the Doppler distortion on the surface waves spectral tails. The general formula (30) for an arbitrary spatial spectra and arbitrary dispersion relation was derived. The case of dispersion relation corresponding to surface gravity waves was considered in detail. The formulae for isotropic (41) and (42) and the strongly anisotropic (45) and (46) spectra were obtained. For the special case of Phillips spectra  $\sim k^{-4}$  we got two asymptotics:  $\omega^{-5}$  in the frequency region close ( $\omega < 2.5\omega_p$  in the case of spectrum peak influence) to the spectral maximum (43) and (47), and  $\omega^{-3}$  for high frequencies (52) and (53).

We numerically calculated the spectra in the wide range of parameters and gave an explanation of the results considering the range of applicability of the assumptions used.

### Acknowledgments

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Also the author wishes to thank creators of GNU Scientific Library [30] for this useful and free software.

### Appendix A. Calculation of the spectra relations and Doppler velocity due to spectral peak's influence

Following [11] or [14] we reformulate our problem using the Hamiltonian formalism.

Let us consider the potential flow of an ideal incompressible fluid of infinite depth and with a free surface. We use standard notations for velocity potential  $\phi(\vec{r}, z, t)$ ,  $\vec{r} = (x, y)$ ;  $v = \nabla\phi$  and surface elevation  $\xi(\vec{r}, t)$ . Fluid flow is irrotational  $\Delta\phi =$

0. The total energy of the system can be represented in the following form

$$H = T + U, \quad T = \frac{1}{2} \int d^2r \int_{-\infty}^{\xi} (\nabla\phi)^2 dz, \quad (A.1)$$

$$U = \frac{1}{2} g \int \xi^2 d^2r, \quad (A.2)$$

where  $g$  — is the gravity acceleration. It was shown [26] that under these assumptions the fluid is a Hamiltonian system

$$\frac{\partial \xi}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \xi}, \quad (A.3)$$

where  $\psi = \phi(\vec{r}, \xi(\vec{r}, t), t)$  is a velocity potential on the surface of the fluid. In order to calculate the value of  $\psi$  we have to solve the Laplas equation in the domain with varying surface  $\eta$ . This is a difficult problem. One can simplify the situation, using the expansion of the Hamiltonian in powers of "steepness". If we limit ourselves only to the linear part

$$H = \frac{1}{2} \int (g\xi^2 + \psi \hat{k} \psi) d^2r. \quad (A.4)$$

Here  $\hat{k}f(\vec{r}) = \hat{F}^{-1}[|k|\hat{F}[f(\vec{r})]]$  — is the linear operator corresponding to multiplying of Fourier harmonics by absolute value of the wavenumber  $k$ . In this case linear dynamical Eq. (A.3) acquire the following form

$$\dot{\xi} = \hat{k}\psi, \quad \dot{\psi} = -g\xi. \quad (A.5)$$

Let us consider velocity field in  $(x, y)$  plane which results from the presence of spectral peak wave. The monochromatic spectral peak wave can be represented as follows

$$\xi_p(\vec{r}, t) = A_p \cos((\vec{k}_p \vec{r}) - \omega_p t).$$

Taking into account one of Hamiltonian Eq. (A.5), velocity field in  $\vec{r} = (x, y)$  plane is given by the formula

$$\vec{v}_p(\vec{r}, t) = \nabla\psi(\vec{r}, t) = -g \int_0^t \nabla\xi_p(\vec{r}, t).$$

Finally, Doppler velocity in the direction of  $\vec{k}_p$  is the following

$$v_p(t) = \frac{gk_p A_p}{\omega_p} \cos((\vec{k}_p \vec{r}) - \omega_p t) = A_p \omega_p \cos((\vec{k}_p \vec{r}) - \omega_p t). \quad (A.6)$$

Fourier harmonics of the real functions  $\psi$  and  $\xi$  have Hermitian symmetry, so it is convenient to introduce the canonical variables  $a_{\vec{k}}$  as shown below

$$a_{\vec{k}} = \sqrt{\frac{\omega_k}{2k}} \eta_{\vec{k}} + i \sqrt{\frac{k}{2\omega_k}} \psi_{\vec{k}}, \quad (A.7)$$

where

$$\omega_k = \sqrt{gk}. \quad (A.8)$$

This is the dispersion relation for the case of gravity waves on the surface of the fluid of infinite depth. Similar formulas can

be derived in the case of finite depth [27]. We should stress, that expression (A.7) is valid for arbitrary dispersion relation.

With these variables the Eq. (A.3) take the following form

$$\dot{a}_{\vec{k}} = -i \frac{\delta H}{\delta a_{\vec{k}}^*}. \quad (\text{A.9})$$

The most important feature of these variables is the absence of Hermitian symmetry; it means that if there is only a monochromatic wave with wave vector  $\vec{k}$ , then only  $a_{\vec{k}}$  harmonics will have nonzero value.

One can introduce spectra

$$\begin{aligned} \langle a_{\vec{k}} a_{\vec{k}'}^* \rangle &= N_{\vec{k}} \delta(\vec{k} - \vec{k}'), \\ \langle a_{\vec{k},\omega} a_{\vec{k}',\omega'}^* \rangle &= N_{\vec{k},\omega} \delta(\vec{k} - \vec{k}') \delta(\omega - \omega'). \end{aligned} \quad (\text{A.10})$$

For the spectrum related with  $N_{\vec{k}}$  Hasselmann Eq. (3) is written.

Using representation of  $\xi_{\vec{k}}$  in terms of  $a_{\vec{k}}$

$$\xi_{\vec{k}} = \sqrt{\frac{k}{2\omega_k}} (a_{\vec{k}} + a_{-\vec{k}}^*),$$

and the definition of the spatial spectrum of the waves, one can get

$$\Phi(\vec{k}) \delta(\vec{k} - \vec{k}') = \langle \xi_{\vec{k}} \xi_{\vec{k}'}^* \rangle = \frac{k}{2\omega} (N_{\vec{k}} + N_{-\vec{k}}) \delta(\vec{k} - \vec{k}'), \quad (\text{A.11})$$

$$\Phi(\vec{k}) = \frac{k}{2\omega} (N_{\vec{k}} + N_{-\vec{k}}). \quad (\text{A.12})$$

In the same way for spatial-temporal spectrum (7)

$$X(\vec{k}, \omega) = \frac{k}{2\omega} (N_{\vec{k},\omega} + N_{-\vec{k},-\omega}). \quad (\text{A.13})$$

In the linear approximation

$$N_{\vec{k},\omega} = N_{\vec{k}} \delta(\omega - \omega_k), \quad (\text{A.14})$$

and relation (A.13) can be written as

$$X(\vec{k}, \omega) = \frac{k}{2\omega} (N_{\vec{k}} \delta(\omega - \omega_k) + N_{-\vec{k}} \delta(\omega + \omega_k)). \quad (\text{A.15})$$

Let's consider two limiting cases of the wave field:

- Spectrum is symmetric on  $\vec{k}$ ;
- Spectrum is unidirectional on  $\vec{k}$ .

For isotropic spectrum

$$N_{\vec{k}} = N_{-\vec{k}}, \quad \Phi(k) = \frac{k}{\omega_k} N_k,$$

$$X_{is}(k, \omega) = \frac{k}{2\omega} N_k (\delta(\omega - \omega_k) + \delta(\omega + \omega_k));$$

$$X_{is}(k, \omega) = \frac{1}{2} \Phi_k (\delta(\omega - \omega_k) + \delta(\omega + \omega_k)). \quad (\text{A.16})$$

For unidirectional spectrum (with direction corresponding to angle  $\theta_0$ )

$$N_{\vec{k}} = N_k \delta(\theta - \theta_0), \quad N_{-\vec{k}} = N_k \delta(\theta - \pi - \theta_0),$$

$$\begin{aligned} X_{an}(\vec{k}, \omega) &= \frac{k}{2\omega} N_k (\delta(\theta - \theta_0) \delta(\omega - \omega_k) \\ &\quad + \delta(\theta - \pi - \theta_0) \delta(\omega + \omega_k)). \end{aligned}$$

We consider positive-frequency part:

$$\begin{aligned} \Phi(\vec{k}) &= \frac{k}{2\omega_k} N_{\vec{k}}, \quad X_{an}(\vec{k}, \omega) = \frac{k}{2\omega} N_{\vec{k}} \delta(\omega - \omega_k); \\ X_{an}(\vec{k}, \omega) &= \Phi(\vec{k}) \delta(\omega - \omega_k). \end{aligned} \quad (\text{A.17})$$

## Appendix B. Calculations of the spectra decompositions. $\lambda < 1/4$

As a first step, we consider a slightly more complex case of isotropic spectrum. Let us perform the following substitution

$$x = \frac{\lambda - \sqrt{\zeta}}{\zeta}. \quad (\text{B.1})$$

In this case in our domain of integration ( $-1 < x < +1$ ) we can obtain the inverse representation

$$\zeta = \frac{1}{2x^2} [1 - \sqrt{1 + 4x\lambda} + 2x\lambda], \quad (\text{B.2})$$

and further

$$\begin{aligned} \zeta(x) &\approx \frac{1}{2x^2} [2\lambda^2 x^2 - 4\lambda^3 x^3 + 10\lambda^4 x^4] \\ &= \lambda^2 - 2\lambda^3 x + 5\lambda^4 x^2. \end{aligned} \quad (\text{B.3})$$

For substitution  $d\zeta = \zeta'(x)dx$  we have to calculate the first derivative

$$\begin{aligned} \zeta'(x) &= -\frac{1}{x^3} - \frac{\lambda}{x^2 \sqrt{1 + 4\lambda x}} + \frac{\sqrt{1 + 4\lambda x}}{x^3} - \frac{\lambda}{x^2} \\ &\approx -\frac{1}{x^3} - \frac{\lambda}{x^2} [1 - 2\lambda x + 6\lambda^2 x^2 - 20\lambda^3 x^3 + 70\lambda^4 x^4] \\ &\quad + \frac{1}{x^3} [1 + 2\lambda x - 2\lambda^2 x^2 + 4\lambda^3 x^3 - 10\lambda^4 x^4 + 28\lambda^5 x^5] \\ &\quad - \frac{\lambda}{x^2} = -2\lambda^3 (1 - 5\lambda x + 21\lambda^2 x^2). \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \Phi(\zeta(x)) &= \Phi(\lambda^2(1 - 2\lambda x + 5\lambda^2 x^2)) \approx \Phi(\lambda^2) \\ &\quad - \frac{\Phi'(\lambda^2)}{1!} 2\lambda^3 x + \frac{\Phi'(\lambda^2)}{1!} 5\lambda^4 x^2 + \frac{\Phi''(\lambda^2)}{2!} 4\lambda^6 x^2. \end{aligned} \quad (\text{B.5})$$

Now we can calculate the integral (41) with Phillips spectrum ( $\Phi(\zeta) = 1/\zeta^4$ )

$$F_{is}(\lambda) = \frac{4}{\pi} \frac{g}{v^3} B \int_{-1}^{+1} \Phi(\zeta(x)) \zeta'(x) K(\sqrt{1-x^2}) dx, \quad (\text{B.6})$$

Taking into account several well known results [24]

$$\begin{aligned} \int_{-1}^{+1} K(\sqrt{1-x^2}) dx &= \frac{\pi^2}{2}, \\ \int_{-1}^{+1} x K(\sqrt{1-x^2}) dx &\equiv 0, \\ \int_{-1}^{+1} x^2 K(\sqrt{1-x^2}) dx &= \int_0^{+1} E(\sqrt{1-x^2}) dx = \frac{\pi^2}{8}, \end{aligned}$$

where  $E$  is the complete elliptic integral of the second kind, one can obtain the final result

$$F_{is}(\lambda) = 4\pi\alpha \frac{v^5}{g^3} \frac{1}{\lambda^5} \left( 1 + \frac{1}{4}\lambda^2 + \dots \right). \quad (B.7)$$

Calculations of the spectrum decomposition for strongly anisotropic case up to expression Eq. (B.6) are just the same. Integral (45) takes the form

$$F_{an}(\lambda) = \frac{2}{\pi} \frac{g}{v^3} B \int_{-1}^{+1} \frac{\Phi(\zeta(x))\zeta'(x)}{\sqrt{1-x^2}} dx, \quad (B.8)$$

The final result is

$$F_{an}(\lambda) = 2\alpha \frac{v^5}{g^3} \frac{1}{\lambda^5} \left( 1 + \frac{1}{2}\lambda^2 + \dots \right). \quad (B.9)$$

### Appendix C. Numerical calculation of the strongly anisotropic spectra at $\lambda > 1/4$

Here we consider Phillips spatial spectrum  $\zeta^{-4}$  but all results are independent with respect to the power of spatial spectrum. We start from (46). Here and further we omit all constants before integral sign

$$F(\lambda) = \int_{\left(\sqrt{\frac{1}{4}+\lambda}-\frac{1}{2}\right)^2}^{+\infty} \frac{\zeta^{-4} d\zeta}{\sqrt{1-\left(\frac{\lambda-\sqrt{\zeta}}{\zeta}\right)^2}}. \quad (C.1)$$

After substitution  $\mu = 1/\sqrt{\zeta}$  one gets

$$F(\lambda) = 2 \int_0^{\left(\sqrt{\frac{1}{4}+\lambda}-\frac{1}{2}\right)^{-1}} \frac{\mu^5 d\mu}{\sqrt{1-(\lambda\mu^2-\mu)^2}}. \quad (C.2)$$

The upper limit of integration gives us a singularity. Usual technique for calculation of this integral consists in excluding of this special point in some way (see for example [28]). In our case one can do this in a very simple and effective way. Our integral can be represented as follows:

$$F(\lambda) = 2 \int_0^{\mu_2} \frac{\mu^5 d\mu}{\sqrt{\lambda(\mu_1-\mu)(\mu-\mu_2)(1+\lambda\mu^2-\mu)}}, \quad (C.3)$$

where we introduced two roots of polynomial under square root sign

$$\mu_1 = \frac{1}{\lambda} \left( \frac{1}{2} - \sqrt{\frac{1}{4} + \lambda} \right),$$

$$\mu_2 = \frac{1}{\lambda} \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \lambda} \right).$$

Using substitution  $v = \sqrt{\mu_2 - \mu}$  one can get

$$F(\lambda) = \frac{4}{\sqrt{\lambda}} \int_0^{\sqrt{\mu_2}} \frac{(\mu_2 - v^2)^5 dv}{\sqrt{(\mu_1 - \mu_2 + v^2)(1 + \lambda(\mu_2 - v^2)^2 - \mu_2 + v^2)}}. \quad (C.4)$$

This integral can be calculated by any standard tool of numerical mathematics, for example Simpson rule [28,29]. In this case even 32 points in domain give us relative error less than  $10^{-6}$  with respect to 16 points. If we would try to calculate our spectrum from the very first formula, to achieve such an accuracy we have to use more than  $10^6$  points in integration domain (because even Simpson rule, usually the method of the third order of accuracy, has only the first order in the vicinity of singular point). With modern computers and such a simple functions as a square root under integration sign it is not very fast but bearable calculation. But if we have to calculate integral of special function (for instance, the complete elliptic integral of the first kind) it will take an enormous amount of time.

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