Jones factorization and Rubio de Francia extrapolation for matrix weights

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In the 1990s, Nazarov, Treil and Volberg introduced a generalization of the scalar Muckenhoupt A_p condition to matrix weights. Let W be a $d \times d$ symmetric, positive definite matrix weight function. For $1 < p < \infty$, we say W is in matrix A_p if

$$[W]_{A_p} = \sup_Q \fint_Q \left(\fint_Q |W^{1/p}(x) W^{-1/p}(y)|_{op}^{p'} \, dy \right)^{p/p'} dx < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$. They showed that the Hilbert transform is bounded on $L^p(W)$, the space of vector-valued functions with norm

$$\|f\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} |W^{1/p}(x) f(x)|^p \, dx \right)^{1/p}.$$

Later, Christ and Goldberg extended this result to all Calderón-Zygmund singular integrals, and a theory of two-weight norm inequalities was introduced by DCU, Isralowitz and Moen.

In the 1990s, Nazarov, Treil and Volberg posed two related problems: extend the Jones factorization theorem and the Rubio de Francia theory of extrapolation, to matrix A_p weights. Since 2011, an important open question has been to prove the A_2 conjecture for matrix weights: it is conjectured that the sharp exponent on the A_p characteristic for matrix weights is the same as for scalar weights, $\max\{1, \frac{1}{p-1}\}$.

In this talk we will discuss the solution of the first two problems. Joint with Marcin Bownik, we have proved exact generalizations of factorization and the sharp constant extrapolation theorem that was used to prove the scalar A_2 conjecture. The proofs required the development of a new family of tools for working with convex-set valued functions. This let us generalize the ideas in the convex-body sparse domination theorem of Nazarov, Petermichl, Treil and Volberg. We defined the analog of the Hardy-Littlewood maximal operator on convex-set valued functions, and proved weighted norm inequalities for this operator on $L^p(W)$ (suitably extended to convex-set valued functions). This allowed us to define a Rubio de Francia iteration algorithm, the fundamental tool in the proof of both factorization and extrapolation.