Math 563, Fall 2016
Assignment 4, due Wednesday, October 12

Hand in the following exercises. Note that there is a difference between an "exercise" and a "problem" in the text, below we refer to the former.

1. Let (X, d) be a metric space and suppose that q is a limit point of X. Suppose that for each $p \in X$, there is an associated measurable function $f_p : \mathbb{R}^d \to \mathbb{C}$ such that

$$\lim_{p \to q} f_p(x) = f(x) \qquad \text{a.e. } x$$

   for some function $f : \mathbb{R}^d \to \mathbb{C}$.

   (a) Prove that f is measurable.

   (b) Suppose there is an integrable function $g : \mathbb{R}^d \to [0, \infty]$ such that $|f_p(x)| \leq g(x)$ for every p in a deleted neighborhood of q. Prove that

$$\lim_{p \to q} \int |f_p(x) - f(x)| \, dx = 0$$

   and hence $\lim_{p \to q} \int f_p = \int f$.

   (c) Let $I \subset \mathbb{R}$ be an open interval and suppose $f(x, t) : \mathbb{R}^d \times I \to \mathbb{C}$ is such that $f(\cdot, t)$ is measurable for all $t \in I$. Suppose further that $\frac{\partial f}{\partial t}(x, t)$ exists for all (x, t) and that there exists an integrable function $g : \mathbb{R}^d \to [0, \infty]$ such that $|\frac{\partial f}{\partial t}(x, t)| \leq g(x)$ for all $t \in I$. Prove that for every $t \in I$,

$$\frac{\partial}{\partial t} \left(\int f(x, t) \, dx \right) = \int \frac{\partial f}{\partial t}(x, t) \, dx.$$

   Make sure to address why both sides of the identity are well defined.

2. Prove the following generalization of the dominated convergence theorem: Suppose $f_n, f : \mathbb{R}^d \to \mathbb{C}$ and $g_n, g : \mathbb{R}^d \to [0, \infty]$ are all integrable functions such that $f_n \to f$, $g_n \to g$ and $|f_n| \leq g$ a.e., and that $\int g_n \to \int g$. Prove that $\int f_n \to \int f$. (Hint: what happens when you apply Fatou's lemma to $g_n \pm f_n$?)

3. Stein-Shakarchi, Exercise #4, Chapter 2.

4. Stein-Shakarchi, Exercise #17, Chapter 2.

5. Stein-Shakarchi, Exercise #19, Chapter 2.

Reading: Stein and Shakarchi, finish Chapter 2 and start Chapter 6. In class, we will go through a "crash course" on Hilbert spaces, covering portions of Chapter 4. We will skip much of the section on $L^1(\mathbb{R}^n)$ and return to it later.