Math 563, Fall 2016
Assignment 5, due Wednesday, November 23

The exercises below do not use the triangle inequality for $L^p(X,\mu)$, so for any $0 < p < \infty$ regard $L^p(X,\mu)$ to be the space of measurable functions f such that $\int_X |f|^p\,d\mu < \infty$ and $\|f\|_p = (\int_X |f|^p\,d\mu)^{1/p}$. In particular, some exercises do not require $p \geq 1$.

1. Let (X,d) be a metric space and $f : X \to \mathbb{R}$. Define for $p \in X$,

$$B_r^*(p) = \{q \in X : 0 < d(p,q) < r\}$$

as the deleted ball of radius r about p. Given a limit point p of X, define the limit superior and limit inferior at p as

$$\liminf_{q\to p} f(q) = \sup_{\delta>0} \inf_{q\in B_\delta^*(p)} f(q) \tag{0.1}$$

$$\limsup_{q\to p} f(q) = \inf_{\delta>0} \sup_{q\in B_\delta^*(p)} f(q) \tag{0.2}$$

The function f is said to be *lower (upper) semicontinuous* at p if

$$\liminf_{q\to p} f(q) \geq f(p) \qquad \left(\limsup_{q\to p} f(q) \leq f(p)\right)$$

respectively. Correspondingly, f is said to be *lower (upper) semicontinuous on* X if it is lower (upper) semicontinuous at all limit points of X.

(a) Explain why the $\inf_{\delta>0}$, $\sup_{\delta>0}$ on the right hand side of (0.1), (0.2) respectively can be replaced by $\lim_{\delta\to 0+}$.

(b) Prove that $\lim_{q\to p} f(q)$ exists if and only if

$$\limsup_{q\to p} f(q) = \liminf_{q\to p} f(q),$$

in which case the limit is equal to this common value.

(c) Prove that f is lower semicontinuous on X if and only if

$$\{p : f(p) > a\}$$

is open in X for every $a \in \mathbb{R}$.

(d) Show that any lower semicontinuous function is Borel measurable.

2. Prove that if $0 < p < q < r \leq \infty$, then $L^q(X,\mu) \subset L^p(X,\mu) + L^r(X,\mu)$; that is, each $f \in L^q(X,\mu)$ can be written as $f = g + h$, the sum of a function in $g \in L^p(X,\mu)$ and a function in $h \in L^r(X,\mu)$.

3. Fix p_0, p_1 with $0 < p_0 < p_1 \leq \infty$. Find examples of functions f on $(0, \infty)$ (with Lebesgue measure), such that $f \in L^p$ if and only if

   (a) $p_0 < p < p_1$

   (b) $p_0 \leq p \leq p_1$

   (c) $p = p_0$

   Hint: consider functions of the form $x^{-a} |\log x|^{-b}$, or possibly piecewise defined functions involving these expressions.

4. Suppose $\mu(X) < \infty$ and $0 < p < q \leq \infty$. Show that $L^q(X, \mu) \subset L^p(X, \mu)$ and

$$\|f\|_p \leq \|f\|_q \, (\mu(X))^{\frac{1}{p} - \frac{1}{q}} .$$

5. (Generalized Hölder inequality) Suppose that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = \frac{1}{r}$$

   with $1 \leq p_j \leq \infty$ for $j = 1, \ldots, k$ and $1 \leq r \leq \infty$. If $f_j \in L^{p_j}(X, \mu)$ for $j = 1, \ldots, k$, then $\Pi_{j=1}^k f_j \in L^r(X, \mu)$ and

$$\|\Pi_{j=1}^k f_j\|_r \leq \Pi_{j=1}^k \|f_j\|_{p_j}$$

   (Hint: using induction, reduce to the case where $k = 2$. Then derive this as a consequence of the usual Hölder inequality.)