# Math 402/502, Spring 2018
# Assignment 10, due Wednesday, April 18

**Problems to hand in**:

1. Wade, Exercise 10.5.5.

   Hint: Consider a proof by contradiction: if $A$ is disconnected, then it is separated by nonempty relatively open sets $U, V$. Argue that since $E$ is connected, we then must have either $E \cap U = \emptyset$ or $E \cap V = \emptyset$. Without loss of generality, assume the former. Since $U = A \cap W$ for some $W$ open in $X$, what does this imply about $\bar{E} \cap W$?

   Recall that for sets $C, D$ that $C \cap D = C$ if and only if $C \subset D$.

2. Wade, Exercise 10.5.10.

   Hint: Detailing the hint in the back of the book, recall that the line segment $L(\mathbf{x}_{k-1}; \mathbf{x}_k)$ between $\mathbf{x}_{k-1}$, $\mathbf{x}_k$ can be parameterized by $s \mapsto (1-s)\mathbf{x}_{k-1} + s\mathbf{x}_k$ for $s \in [0,1]$. Consequently, polygonal paths as given in (a) can be parameterized as the image of the continuous function $f : [0, N] \to \mathbb{R}^n$ defined by taking $f(s) = (k-s)\mathbf{x}_{k-1} + (s-k+1)\mathbf{x}_k$ on $[k-1, k]$. Admittedly the domain of this is not $[0,1]$ as in the book's hint, but the salient feature of $f$ is that the domain is an interval and hence connected.

3. Wade, Exercise 10.6.5.

4. Wade, Exercise 10.6.7.

   Tip: Use Theorems 10.52 and 10.63 liberally.

5. Wade, Exercise 11.1.3.

   Hint: Argue that $f(\mathbf{x}) = f(\mathbf{a})$ for all $x \in B_r(a)$. Writing the difference $f(\mathbf{x}) - f(\mathbf{a})$ in the following way will allow you to make efficient use of the hypothesis

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{a}) = \big(f(x_1, x_2, \ldots, x_n) - f(a_1, x_2, \ldots, x_n)\big) + \\ \big(f(a_1, x_2, \ldots, x_n) - f(a_1, a_2, \ldots, x_n)\big) + \cdots \\ \cdots + \big(f(a_1, \ldots, a_{n-1}, x_n) - f(a_1, \ldots, a_{n-1}, a_n)\big). \end{aligned}$$

**On your own**: Wade 10.5.2, 10.5.4, 10.5.8, 10.6.1, 10.6.6a, ~~11.1.5~~, ~~11.1.6~~ as well as the following problems (See the back page as well):

1. Let $(X, \rho)$ be any discrete space (i.e. a nonempty set endowed with the metric $\rho(x,y) = 1$ when $x \neq y$ and $\rho(x,y) = 0$ when $x = y$). Prove that a nonempty subset $E \subset X$ is connected if and only if $E = \{a\}$ (i.e. $E$ is a "singleton").

2. Let $E \neq \emptyset$ be a subset of a metric space $(X, \rho)$. In Assignment 8, you considered the function $\rho_E(x) = \inf_{z \in E} \rho(x, z)$ and proved that $\rho_E(x) = 0$ if and only if $x \in \bar{E}$.

   (a) Prove that $\rho_E$ defines a uniformly continuous function on $X$, by showing that
   $$|\rho_E(x) - \rho_E(y)| \leq \rho(x, y)$$
   Hint: For any $z \in X$, $\rho_E(x) \leq \rho(x, z) \leq \rho(x, y) + \rho(y, z)$. Use this to show $\rho_E(x) \leq \rho(x, y) + \rho_E(y)$.

   (b) In this part and the next, let $A, B \subset X$ be disjoint nonempty closed sets. Show that $f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$ defines a continuous function on $X$ whose range lies in $[0, 1]$. Moreover, show that $f(p) = 0$ if and only if $p \in A$ and that $f(p) = 1$ if and only if $p \in B$.

   (c) Show that there exists disjoint open sets $V, W \subset X$ such that $A \subset V$, $B \subset W$.

3. Important Review: Let $X, Y$ be nonempty sets. Prove the following properties concerning a function $f : X \to Y$, and subsets $E, E_\alpha \subset Y$, $G \subset X$. These are basic consequences of well-defined functions and hence do not rely any continuity or metric properties of $f$ or $X, Y$.

   (a) $f^{-1}(E^C) = [f^{-1}(E)]^C$

   (b) $f(f^{-1}(E)) \subset E$

   (c) $G \subset f^{-1}(f(G))$

   (d) $f^{-1}(\cup_\alpha E_\alpha) = \cup_\alpha f^{-1}(E_\alpha)$

   (e) $f^{-1}(\cap_\alpha E_\alpha) = \cap_\alpha f^{-1}(E_\alpha)$

   Find examples of functions $f$ such that equality in 3b and 3c fails to hold. Prove that equality in 3b holds whenever $f$ is surjective and equality in 3c holds whenever $f$ is injective.

**Reading:** Wade, sections 10.6, 11.1, 11.2.