Math 402/502, Spring 2018
Assignment 3, due Wednesday, February 7

Problems to hand in:

1. Wade, Exercise 5.2.6.
2. Wade, Exercise 5.3.7.

   Note: The purpose of this exercise is to *define* the natural logarithm from principles in calculus, then derive some of the key properties of this function from the definition. For example, even though it is well-known that $\log(xy) = \log x + \log y$, in (d) you are to derive that property from the given definition, using a change of variables to show

   $$\int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_1^y \frac{dt}{t}.$$

3. The function $f(x) = x/|x|$ satisfies $f'(x) = 0$ for every $x \neq 0$. This leads to the following apparent contradiction to Theorem 5.28(ii) (second part of the Fundamental theorem of calculus):

   $$0 = \int_{-1}^{1} f'(x)\,dx = f(1) - f(-1) = 1 - (-1) = 2.$$

   Explain why this is not actually a contradiction.

4. Let $I \subset \mathbb{R}$ be an interval and fix $a \in I$. Suppose that $f : I \to \mathbb{R}$ is $n+1$-times continuously differentiable, that is, $f^{(k)}(x)$ (the k-th derivative of $f(x)$) exists for all $1 \leq k \leq n+1$ and defines a continuous function. Prove *Taylor's theorem with integral remainder*: for each $x \in I$, $f(x)$ satisfies

   $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

   with $R_n(x)$ given by

   $$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$$

   Hint: The statement is amenable to an induction argument, and the $n = 0$ case is just the Fundamental Theorem of Calculus. Use integration by parts and that

   $$-\frac{1}{n!}\frac{d}{dt}(x-t)^n = \frac{1}{(n-1)!}(x-t)^{n-1}.$$

See the back page for the "On your own" problems and reading assignments.

On your own (i.e. do not hand these in for a grade): Wade 5.2.5, 5.2.10 (along with 3.1.8 if needed), 5.3.5, 5.4.2(a,b) as well as the following problem:

Using the conventions $\int_a^a f(x)\,dx = 0$ and $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ when $b < a$, show that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

regardless of the ordering of a, b, c *in the real number line.* Note that by elementary combinatorics, there are $3! = 6$ possible choices for the ordering of a, b, c (e.g. $a < b < c,\ a < c < b,\ b < a < c, \dots$).

Reading: Wade, finish 5.2, complete 5.3, review definition of improper integral in 5.4.