Math 402/502, Spring 2018
Assignment 5, due Wednesday, February 28

Problems to hand in:

1. Decide if the following sequences of functions converge uniformly, pointwise but not uniformly, or neither on the given set E:

   (a) $f_n(x) = x/n$, $E = (0, 1)$.

   (b) $f_n(x) = \sin(x/n)$, $E = \mathbb{R}$.

   (c) $f_n(x) = \sin(n\pi x)$, $E = \mathbb{Q}$.

   (d) $f_n(x) = x^2 + \frac{x}{n}$, $E = [0, \infty)$.

2. (a) Wade, Exercise 7.1.3.

   (b) Suppose f_n, g_n are sequences of functions on a set $E \subset \mathbb{R}$. Show that if f_n is a uniformly bounded sequence and $g_n \to 0$ uniformly on E, then $f_n g_n \to 0$ on E.

3. Suppose $f_n \to f$ and $g_n \to g$ uniformly on a set $E \subset \mathbb{R}$. It is not hard to check that $f_n g_n \to fg$ pointwise by standard limit properties (do this on your own, see Theorem 2.15(iii) in the text).

   (a) Show that if f, g are bounded functions on E, then $f_n g_n \to fg$ uniformly on E.
   Hint: While in general, you cannot deduce that the full sequence $\{g_n\}_{n=1}^\infty$ is uniformly bounded, it is possible to argue there exists N such that
   $$\sup_{x\in E} |g_n(x)| \leq \sup_{x\in E} |g(x)| + 1 \qquad \text{for } n \geq N,$$
   so that the sequence of functions $\{g_n\}_{n=N}^\infty$ is uniformly bounded. Then write $f_n g_n - fg = (f_n - f)g_n + f(g_n - g)$ and appeal to Exercise 2 above (which will apply for $n \geq N$).

   (b) Show that in general, part (a) is false if f, g are unbounded.
   Hint: Let $E = [0, \infty)$ and try $f_n(x) = x$, $g_n(x) = x + \frac{1}{n}$.

   (c) Suppose further that there exists a uniform constant $c > 0$ such that $f_n(x) \geq c$ for every $x \in E$ and every n. Show that $1/f_n \to 1/f$ uniformly on E.

   (d) Show that in general, part (c) is false if f_n is merely assumed to be positive on E, i.e. $f(x) > 0$ for every $x \in E$, but there is no uniform lower bound for each f_n.
   Hint: Consider 1a above.

4. Wade, Exercise 7.1.6.

   Hint: Adapt the proof of Theorem 7.9.

See the back page for the "On your own" problems and reading assignments.

On your own (i.e. do not hand these in for a grade): Wade 7.1.1, 7.1.7 as well as the following problems:

1. Consider the sequence $f_n(x) = x^n(1-x)$, $E = [0,1]$ in the context of Exercise 1 above.

   Hint: Find the maximum of $x^n(1-x)$ on $[0,1]$ for each n.

2. Use the fundamental theorem of calculus and that $(\sin x)' = \cos x$ to show that for any $x \in \mathbb{R}$, $|\sin x| \leq |x|$. Then use this to show that $f_n(x) = \sin(nx)$ converges uniformly on any interval $[0,b]$ with $b > 0$.

3. Suppose $f_n \to f$ and $g_n \to g$ uniformly on a set $E \subset \mathbb{R}$. Show that $f_n + g_n \to f + g$ uniformly and $\alpha f_n \to \alpha f$ uniformly on E.

Reading: Wade, sections 7.1-7.2.