Math 402/502, Spring 2018
Assignment 6, due Wednesday, March 7

Problems to hand in:

1. Wade, Exercise 7.2.3.
2. Wade, Exercise 7.2.5.

   Hint: Use the result from the "On your own" exercise in Assignment 5 that $|\sin x| \le |x|$. You may find it helpful to use that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.
3. Show that the series $\sum_{k=0}^{\infty} \frac{x^2}{(1+x^2)^k}$ converges pointwise to 0 if $x = 0$ and to $1 + x^2$ if $x \neq 0$. Is the convergence uniform on $[-1, 1]$? Is the convergence uniform on $[b, \infty)$ if $b > 0$? Justify your answers.
4. Wade, Exercise 7.3.1. Fully justify your answer. Note that the problem asks for the *interval of convergence* rather than just the radius, so the back of the book should only be taken as a hint.
5. Wade, Exercise 7.3.5. More precisely you should be able to show that the radius of convergence satisfies $R \ge 1$ here.

On your own (i.e. do not hand these in for a grade): Wade 7.2.2, 7.3.2, 7.3.7 as well as the following problems:

1. Prove that $f(x) = \sum_{k=1}^{\infty} \frac{x^2}{k}$ defines a continuous function on $[-1, 1]$.
2. Let $b > 0$. Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x^2 + k)}{k^2}$$

   is uniformly convergent on $[0, b]$, but is not absolutely convergent for any $x \ge 0$.
3. After solving Exercise #1 above, consider the following:

   Recall the function $L : (0, \infty) \to \mathbb{R}$ defined in Exercise 5.3.7 of the text (the natural logarithm, see Assignment 3). There we saw that $L'(y) = \frac{1}{y} > 0$ for every $y > 0$ so that L is strictly increasing and hence injective (a.k.a. one-to-one).

   (i) Use that $L(y) \to \pm\infty$ as $y \to \pm\infty$ and the intermediate value theorem to show that L is surjective (a.k.a. onto).

   (ii) Given the previous part, $L : (0, \infty) \to \mathbb{R}$ is now a bijection and hence it has an inverse map $L^{-1} : \mathbb{R} \to (0, \infty)$. Hence by the inverse function theorem (Theorem 4.33 in Wade or Theorem 10.4.2 in Tao), the inverse function $L^{-1}(x)$ is differentiable and $(L^{-1})'(x) = \frac{1}{L'(y)}$ whenever $L(y) = x$. Use this and part (c) to show that E is the inverse function of L.

(iii) Prove that

$$L(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} \qquad \text{for } |x| < 1,$$

by justifying term by term integration of $\frac{1}{1+t} = \sum_{k=0}^{\infty} (-1)^k t^k$.

Reading: Wade, sections 7.2-7.3.