Math 402/502, Spring 2018
Assignment 7, due Wednesday, March 21

Problems to hand in:

1. (Wade Exercise 7.3.10) Suppose that $a_k \downarrow 0$ as $k \to \infty$. Prove that given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \sum_{k=0}^{\infty} (-1)^k a_k (x^k - y^k) \right| < \varepsilon$$

for all $x, y \in [0,1]$ which satisfy $|x - y| < \delta$. In other words, show that

$$f(x) = \sum_{k=0}^{\infty} (-1)^k a_k x^k$$

defines a *uniformly* continuous function on $[0,1]$ by Abel's Theorem. Exercise 7.1.6 may be useful here.

2. Let $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ be the function considered in Exercise 7.2.3. Use Corollary 7.34 and the binomial formula to show that $E(x)E(y) = E(x+y)$ for each $x, y \in \mathbb{R}$.

   Note: As shown in Example 7.45, $E(x) = e^x$ and this is of course a familiar property of exponents. The goal of the problem is therefore to derive the same property independently using results on multiplication of power series.

3. Wade, Exercise 7.4.8.

4. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Prove that f has derivatives of all orders on $\mathbb{R}$ with $f^{(n)}(0) = 0$ for all n. You may use all usual properties of the exponential functions, including $\frac{d}{du} e^u = e^u$ and $\lim_{u \to \infty} e^u = \infty$, $\lim_{u \to -\infty} e^u = 0$,

Hint: first prove that for every $n \in \mathbb{N}$, the function $f^{(n)}(x)$ on $(0, \infty)$ takes the form $p_{3n}(\frac{1}{x})e^{-1/x^2}$ for some p_k is a polynomial of degree k. Then show $\lim_{x \to 0+} \frac{1}{x^k} e^{-1/x^2} = 0$ for every $k \in \mathbb{N}$.

Note: This is a slight modification of the function in Remark 7.41, the only difference being the definition of the function for $x \leq 0$. The version in this exercise arises in some applications, such as the construction of "bump functions".

On your own (i.e. do not hand these in for a grade): Wade 7.3.6, 7.3.9, 7.4.1, 7.4.2(a,b,c), 7.4.6 as well as the following problem:

1. Suppose f, g are analytic functions on an open interval (a, b). Prove that $f + g$ and fg define analytic functions on (a, b).

Reading: Wade, sections 7.4, 8.1, 8.2, 10.1.
Important note: Sections 8.1 and 8.2 should more or less be review, so we will not spend time on them in class.