Math 402/502, Spring 2018
Assignment 8, due Wednesday, April 4

Note: If a metric on $\mathbb{R}^n$ is not specified, you can take it to be the usual metric $|x-y| = (\sum_{j=1}^n |x_j - y_j|^2)^{1/2}$. When $n=1$, this of course is just the absolute value of $x-y$.

Problems to hand in:

1. Wade, Exercise 10.2.2.
2. (Wade, Exercise 10.3.6) Let (X, ρ) be a metric space and $E \subset X$. Prove that $x \notin E^\circ$ if and only if $B_r(x) \cap E^c \neq \emptyset$ for every $r > 0$.

   Tip: You can consider proving the logically equivalent statement that $x \in E^\circ$ if and only if there exists $r > 0$ such that $B_r(x) \subset E$. This uses that the statement $p \Leftrightarrow q$ is equivalent to the statement $\neg p \Leftrightarrow \neg q$ (essentially the result of taking the contrapositives of both $p \Rightarrow q$ and $p \Leftarrow q$).
3. Wade, Exercise 10.3.7(a,b).
4. Wade, Exercise 10.3.8.

   Note: This exercise ties in with our discussion of relatively open and closed sets in class. In particular, "open/closed in Y" means "open/closed in the metric space (Y, ρ)" respectively.
5. Let (X, ρ) be a metric space and E be a nonempty subset of X. Define the distance from a point $a \in X$ to E as

   $$\rho_E(a) := \inf \{\rho(a, x) : x \in E\}.$$

   Prove that $\rho_E(a) = 0$ if and only if $a \in \bar{E}$.

   Note: This is a stronger statement than the one in Exercise 10.3.5.

On your own: Wade 10.2.1, 10.2.3, 10.2.6, 10.2.8, 10.3.1, 10.3.2, 10.3.4 as well as the following problems:

1. Consider $\mathbb{R}$ with the usual metric and suppose $E \subset \mathbb{R}$ is bounded above. Prove that $\sup E \in \bar{E}$.
2. (Wade, Exercise 10.2.5) Prove Theorem 10.26.

   Note: Once part (ii) is established, you can make quick work of parts (iii)-(v) by appealing to standard limit properties of sequences in $\mathbb{R}^n$. In turn this can often be reduced to limit properties in $\mathbb{R}$ by Exercise 10.1.3b.
3. Let (X, ρ) be a metric space and $E \subset X$. Prove that the complement of E° is the closure of the complement of E. In other words, show that $(E^\circ)^c = \overline{E^c}$.

Reading: Wade, sections 10.2, 10.3, start 10.4.