Math 511, Spring 2018
Assignment 10, Due Wednesday, April 18

Exercises to hand in:

1. Let ψ be the following parameterization of the sphere of radius 1:

$$\psi(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

Which of the following 2-forms on $T_{(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2})}\mathbb{R}^3$ determine the same orientation on the sphere as that induced by ψ? Which do not determine an orientation at all?

   (a) $\alpha = dx \wedge dy + 2dy \wedge dz$

   (b) $\beta = dx \wedge dy - 2dy \wedge dz$

   (c) $\gamma = dx \wedge dz$

2. Let ω be the differential 1-form on $\mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

Let C be any circle centered at the origin, oriented counterclockwise. Show that $\int_C \omega = 2\pi$.

3. Integrate the 2-form

$$\omega = \frac{1}{x}dy \wedge dz - \frac{1}{y}dx \wedge dz$$

over the top half of the unit sphere using two different parametrizations:

   (a) Spherical coordinates as used above

   (b) Polar coordinates $\phi(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$

Assume that the surface is oriented by the 2-form $dx \wedge dy$ on $T_{(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2})}\mathbb{R}^3$.

Note: Strictly speaking, this is a poorly stated problem since ω is not defined on the $x = 0$ plane and the $y = 0$ plane and the upper hemisphere intersects those two sets. However, if you ignore this technicality and set up the integrals for ω using the proper formulae, you should obtain reasonable answers nonetheless. In other words, the problem is well-defined if interpreted improperly.

4. Let S be the surface obtained by rotating the graph of $z = x^3$, $0 \le x \le 1$, about the z-axis. Integrate the 2-form $\omega = y\,dx \wedge dz$ over S. Assume that S is oriented by the 2-form $dy \wedge dz$ at the point $(\frac{1}{2}, 0, \frac{1}{8})$. (Hint: Use cylindrical coordinates to parametrize S.)

5. Suppose M is a parameterized surface with $\phi : R \to M \subset \mathbb{R}^m$ a bijection and $R \subset \mathbb{R}^n$ an open Jordan region. Suppose further that $\phi'(p)$ has rank n. Define the tangent space to M at $\phi(p)$ to be the subspace of $T_{\phi(p)}\mathbb{R}^m$ spanned by the linearly independent set $\frac{\partial \phi}{\partial x_1}(p), \ldots, \frac{\partial \phi}{\partial x_n}(p)$. Finally, suppose $n < m$ and let ω be the n-form $\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$.

   Show that if for every $\phi(p) \in M$, the standard basis vector $\mathbf{e}_{n+1}$ is in the tangent space to M at $\phi(p)$, then $\int_M \omega = 0$, regardless of how the surface is oriented.

On your own: Choose supplementary exercises in Chapter 4 of Bachman as appropriate. Also the following problem:

Let ω be the differential 2-form on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ defined by

$$\omega = \frac{x}{(x^2+y^2+z^2)^{3/2}} dy \wedge dz - \frac{y}{(x^2+y^2+z^2)^{3/2}} dx \wedge dz + \frac{z}{(x^2+y^2+z^2)^{3/2}} dx \wedge dy.$$

Let S be the sphere of radius ρ, centered at the origin, oriented outward. Show that for any choice of $\rho > 0$, $\int_S \omega = 4\pi$.

Reading: Bachman, 4.1-4.5, 7.1.