Math 511, Spring 2018
Assignment 11, Due Wednesday, April 25

Definition: An n-form ν is said to be *closed* if $d\nu = 0$. It is said to be *exact* if $\nu = d\eta$ for some $n-1$ form η. Since $d(d\eta) = 0$, any exact form is closed, however in exercise 2 below you will see that there exist closed forms which are not exact.

Exercises to hand in:

1. Let $P, Q : U \to \mathbb{R}$ and $f, g, h : V \to \mathbb{R}$ be C^∞ functions on open sets $U \subset \mathbb{R}^2$, $V \subset \mathbb{R}^3$ respectively.

   (a) Let $\omega = P(x,y)\,dx + Q(x,y)\,dy$ be defined on U. Prove that
   $$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

   (b) Let $\omega = f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$. Show that
   $$d\omega = a(x,y,z)\,dy \wedge dz - b(x,y,z)\,dx \wedge dz + c(x,y,z)\,dx \wedge dy$$
   where the coefficient functions a, b, c satisfy $\mathbf{curl}\langle f, g, h\rangle = \langle a, b, c\rangle$.

   (c) Now let ω be the differential 2-form
   $$\omega = f(x,y,z)\,dy \wedge dz - g(x,y,z)\,dx \wedge dz + h(x,y,z)\,dx \wedge dy.$$
   Compute $d\omega$ in terms of f, g, h and simplify your answer so that it takes the form $d\omega = c(x,y,z)\,dx \wedge dy \wedge dz$ for some real function c.

2. Let ω be the differential 1-form on $\mathbb{R}^2 \setminus \{(0,0)\}$
   $$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

   (a) Show that $d\omega = 0$, that is, ω is closed.

   (b) Now let $U = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$, the slit plane formed by removing the non-positive x-axis. Prove that for suitable choices of the branches of $\tan^{-1}$ and $\cot^{-1}$, the following function f defines a $\mathcal{C}'(U)$ function such that $df = \omega$ on U:
   $$f(x,y) = \begin{cases} \tan^{-1}(\frac{y}{x}), & x \neq 0, \\ \cot^{-1}(\frac{x}{y}), & y \neq 0. \end{cases}$$

   (c) Show that if $g \in \mathcal{C}'(U)$ is any other function satisfying $dg = \omega$, then $g - f$ is constant on U.

   Hint: Use Exercise 9, Chapter 9 in Rudin.

   (d) Show that ω is not exact on $\mathbb{R}^2 \setminus \{(0,0)\}$, that is, there is no $g \in \mathcal{C}'(\mathbb{R}^2 \setminus \{(0,0)\})$ such that $dg = \omega$.

   Hint: If $\omega = dg$ was exact on $\mathbb{R}^2 \setminus \{(0,0)\}$, then $g(x,y) = f(x,y) + c$ for $(x,y) \in U$ with c constant. Then argue that if $x < 0$, the one sided limits $\lim_{y \to 0\pm} g(x,y)$ yield different answers.

3. In this exercise you will consider properties of the spherical coordinate transformation

$$\begin{aligned} x &= \rho \sin\phi \cos\theta, \\ y &= \rho \sin\phi \sin\theta, \\ z &= \rho \cos\phi. \end{aligned}$$

   (a) Derive the following expressions for dx, dy, dz in terms of $d\rho, d\phi, d\theta$

$$\begin{aligned} dx &= \sin\phi\cos\theta\, d\rho + \rho\cos\phi\cos\theta\, d\phi - \rho\sin\phi\sin\theta\, d\theta, \\ dy &= \sin\phi\sin\theta\, d\rho + \rho\cos\phi\sin\theta\, d\phi + \rho\sin\phi\cos\theta\, d\theta, \\ dz &= \cos\phi\, d\rho - \rho\sin\phi\, d\phi. \end{aligned}$$

   (b) Derive expressions for $dy \wedge dz$, $dx \wedge dz$, and $dx \wedge dy$ in terms of $d\phi \wedge d\theta$, $d\rho \wedge d\theta$, $d\rho \wedge d\phi$.

   (c) Derive the identity $\rho^2 \sin\phi\, d\rho \wedge d\phi \wedge d\theta = dx \wedge dy \wedge dz$.

Reading: Bachman, 5.1-5.5.
On your own: Practice computing exterior derivatives as in exercise 5.12 in Bachman and also the following exercises:

1. In this exercise you will consider properties of the polar coordinate transformation $(x, y) \mapsto (r, \theta)$.

   (a) Derive the expression

$$dr = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy.$$

   Moreover, show that $d\theta = \omega$ where ω is as in Exercise 2.

   (b) Use algebra to prove that the pulling back the inverse transformation yields

$$\begin{aligned} dx &= \cos\theta\, dr - r\sin\theta\, d\theta \\ dy &= \sin\theta\, dr + r\cos\theta\, d\theta \end{aligned}$$

   Note: You could use calculus instead of algebra here, but it may be more work.

   (c) Use the above to prove the identity $r\, dr \wedge d\theta = dx \wedge dy$.

2. Recall that if ω, λ are k and m-forms respectively, then $d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^k \omega \wedge d\lambda$.

   (a) Show that if ω and λ are closed, then so is $\omega \wedge \lambda$.

   (b) Show that if ω is closed and λ is exact, then $\omega \wedge \lambda$ is exact.

3. Let $m \in \mathbb{N}$ be and consider the $n-1$-form η on $\mathbb{R}^n \setminus \{0\}$ defined by

$$\eta = \sum_{i=1}^{n} \frac{x_i}{|x|^m} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

   where $\widehat{dx_i}$ denotes that the form dx_i is removed from the wedge product. Compute $d\eta$ and show that η is closed if and only if $m = n$.