Math 511, Spring 2018
Assignment 12, Due Wednesday, May 2

Exercises to hand in:

1. Show that the following sets in $\mathbb{R}^3$ are 2-cells. Then find their respective boundaries, assuming that the orientation is induced by parameterization you find in each case. Do there exist differential 1-forms ω such that $\int_\sigma d\omega \neq 0$? Justify your answers.

   (a) The portion of the cone $z^2 = x^2 + y^2$ between the planes $z = 0$ and $z = 1$.

   (b) The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where a, b, c are all positive constants.

   (c) The portion of the sphere $x^2 + y^2 + z^2 = 4$ lying between the $z = 0$ and $z = 1$ planes.

2. Suppose $f(z) \geq 0$ is a nonnegative C^∞ function on $[0, 1]$. Let $\sigma \subset \mathbb{R}^3$ be the 2-cell defined by

$$\phi(\theta, z) = (f(z)\cos(2\pi\theta), f(z)\sin(2\pi\theta), z), \qquad (\theta, z) \in [0, 1]^2.$$

   Assume σ is equipped by the orientation determined by ϕ. Recall that we consider the 1-chain $\partial\sigma$ to be empty if $\int_{\partial\sigma} \omega = 0$ for every differential 1 form ω on $\partial\sigma$. Prove that $\partial\sigma = \emptyset$ if and only if $f(1) = f(0) = 0$.

3. Bachman, Problem 6.25 (**2nd Edition**). I did not type this out since it is lengthy to state and the figure is very helpful.

   Note: For the last part, it is possible to check your answer directly even though you are not asked to do so.

4. Calculate the volume of a ball of radius 1 in $\mathbb{R}^3$ by integrating some 2-form ω over the unit sphere and using the generalized Stokes' Theorem.

5. Suppose $\Omega \subset \mathbb{R}^3$ is a region whose boundary is a smooth orientable surface S for which the divergence theorem applies.

   (a) Given a vector field $\mathbf{F}(x, y, z)$ and a scalar function $\phi(x, y, z)$ prove the identity

$$\nabla \cdot (\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi\nabla \cdot \mathbf{F}.$$

   (b) Prove Green's first identity, show that if $\phi(x, y, z)$, $\psi(x, y, z)$ are scalar functions which are C^2 in a neighborhood of Ω, then

$$\int_\Omega \phi\nabla^2\psi \, dV = \int_S \phi\nabla\phi \cdot \mathbf{n} \, dS - \int_\Omega \nabla\phi \cdot \nabla\psi \, dV$$

   where $\mathbf{n}$ is the unit normal pointing outward from S and $\nabla^2\psi = \nabla \cdot (\nabla\psi)$ denotes the Laplacian of ψ.

   (c) A vector field $\mathbf{F}(x, y, z)$ is said to be *conservative* if there exists a scalar function ϕ such that $\mathbf{F}(x, y, z) = \nabla\phi(x, y, z)$. Suppose $\mathbf{F}$, $\mathbf{E}$ are infinitely differentiable, conservative vector fields on $\mathbb{R}^3$ such that $\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{E}$ throughout Ω and that $\mathbf{F} \cdot \mathbf{n} = \mathbf{E} \cdot \mathbf{n}$ on S. Show that $\mathbf{F} = \mathbf{E}$ throughout Ω.

Reading: Bachman, Chapters 6.1-6.3 (2nd Edition). Also consult a textbook in multivariable calculus to review the derivation of the formulas for line and surface integrals of vector fields.

See back page for "On your own" problems.

On your own: Select exercises in Chapter 6 of Bachman, 2nd Ed. as appropriate. Complete the following vector calculus exercises:

1. In prior assignments, you saw that the differential 1-form ω on $\mathbb{R}^2 \setminus \{(0,0)\}$ defined by

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

   satisfies $d\omega = 0$ but that $\int_C \omega = 2\pi$ for any counterclockwise oriented circle C centered at the origin.

   (a) Discuss the implications this has for the generalized Stokes' theorem.

   (b) Show that if C is any simple closed curve which enclosing the origin in a counterclockwise fashion, then $\int_C \omega = 2\pi$. This means that there exists a C^∞ parameterization $\phi : [0,1] \to \mathbb{R}^2$ of C such that $\phi(0) = \phi(1)$ (closed) but that ϕ is injective on $[0,1)$ (simple). You may assume that Green's theorem applies to the region enclosed by C.

   (c) Now suppose $S \subset \mathbb{R}^3$ is an arbitrary closed surface enclosing the origin, i.e. $S = \partial E$ for some region $E \subset \mathbb{R}^3$ with $0 \in E^\circ$. Assume the divergence theorem applies to E and S. Show that similarly,

$$\int_S \frac{x}{(x^2+y^2+z^2)^{3/2}} dy \wedge dz - \frac{y}{(x^2+y^2+z^2)^{3/2}} dx \wedge dz + \frac{z}{(x^2+y^2+z^2)^{3/2}} dx \wedge dy = 4\pi.$$

2. Let Π be a plane in $\mathbb{R}^3$ with unit normal $\mathbf{n}$ containing the point $P_0(x_0, y_0, z_0)$. For each $r > 0$, let S_r be the disc in Π centered at P_0 of radius $r > 0$, that is $S_r = \Pi \cap B_r(P_0)$. Also, let C_r denote an oriented circle of radius r which forms the boundary of S_r. Show that if F is a continuously differentiable vector field, then there exists a constant $\alpha = \pm 1$ such that

$$\alpha \left(\nabla \times \mathbf{F}\right)(P_0) \cdot \mathbf{n} = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{C_r} \mathbf{F} \cdot \mathbf{T}\, ds.$$

3. (a) Given $0 < s < t < \infty$, define $\sigma = \{(x,y) \in \mathbb{R}^2 : s^2 \le x^2 + y^2 \le t^2\}$. Show that σ is a 2-cell.

   (b) Suppose $P(x,y)$, $Q(x,y)$ are continuously differentiable functions on $\mathbb{R}^2 \setminus \{(0,0)\}$ satisfying $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{1}{\sqrt{x^2+y^2}}$. Find a relationship of the form

$$\int_{C_t} P\,dx + Q\,dy = f(s,t) + \int_{C_s} P\,dx + Q\,dy$$

   where C_s, C_t are the circles of radius s, t about the origin and $f(s,t)$ is some expression depending on s and t.

4. Let ω, λ be k and m-forms respectively. Prove the following integration by parts formula for $m+k$-chains C:

$$\int_C d\omega \wedge \lambda = \int_{\partial C} \omega \wedge \lambda + (-1)^{k+1} \int_C \omega \wedge d\lambda.$$