Math 511, Spring 2018
Assignment 1, due Wednesday, January 24

1. Suppose $f : (a, b) \to \mathbb{R}$ is an infinitely differentiable (a.k.a. C^∞) function and that there exists $M > 0$ such that $|f^{(n)}(x)| \leq M^n$ for all $x \in (a, b)$. Let $x_0 \in (a, b)$. Prove that for all $x \in (a, b)$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

   Note: Recall that it is generally false that a C^∞ function is equal to its Taylor series. For example, the function in II.2 of Assignment #8 in Math 510 cannot be given by its Taylor series at the origin. The point of the exercise is for you to prove a sufficient condition for a C^∞ function to be equal to its Taylor series.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous and satisfies $f(x + y) = f(x)f(y)$ for every $x, y \in \mathbb{R}$. Prove that if f is not the zero function, then $f(x) = e^{cx}$ for some constant $c \in \mathbb{R}$ (and hence $f(x) = b^x$ for some $b > 0$).

   Note: Make sure you justify why f cannot take on any negative values. You can use I.1b from Assignment #7 in Math 510 if you like.

3. Prove that the series

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log \left(1 + \frac{1}{k} \right) \right)$$

   is convergent. The sum γ of the series is called the *Euler-Mascheroni constant.* Conclude that if $s_n = \sum_{k=1}^{n} \frac{1}{k}$ is the n-th partial sum of the harmonic series, then

$$\gamma = \lim_{n\to\infty} s_n - \log n.$$

On your own: Rudin, Chapter 8, exercises 4 (try to work as many as you can without using l'Hôpital's rule), 5, 7, 8 and the following problem:

1. Suppose f is analytic on (a, b) and that $\int_c^d |f(x)|\, dx = 0$ for some nonempty interval $(c, d) \subset (a, b)$. Show that f is identically zero on (a, b).

Reading: Rudin, Chapter 8, sections on exponential and logarithmic functions, trigonometric functions, and Fourier series. Also the section "The number e" in Chapter 3 of Rudin.