Math 511, Spring 2018
Assignment 2, due Wednesday, January 31

In all exercises, use formulas (62) and (63) to define Fourier series (as opposed to sines and cosines).

1. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and 2π-periodic, then its Fourier series converges uniformly to f.

   Hint: Using formula (62) for the Fourier coefficients, prove that if c_n, b_n denote the Fourier coefficients of $f(x)$ and $f'(x)$ respectively then $c_n = \frac{b_n}{in}$ for $n \neq 0$. How can you use #7 in Chapter 3 of Rudin?

2. Rudin, Chapter 8, #15

   Note: This problem is long! See the below for a hint on the very first part. In Math 510, we did not cover the proof of Theorem 7.26 in class, rather we viewed the polynomial approximation theorem as a special case of the more general Stone-Weierstrass theorem. You may wish to review the proof in the text or gather inspiration from the methods in Math 510, Assignment 11, problems I.3 and the written "on your own" problem there (or even the uncollected Exercise #11, Chapter 8 below).

3. Rudin, Chapter 8, #19

On your own: Rudin, Chapter 8, exercises 11, 12, 13 and the following problem:

1. Prove the following trigonometric identities using the definitions from class $\cos x = Re(E(ix))$, $\sin x = Im(E(ix))$:

   (a) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

   (b) $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Reading: Rudin, Chapter 8, sections on Fourier series and the Gamma function.

Hint on the identity $K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}$: Use that

$$D_n(x) = \frac{\sin((n+\frac{1}{2})x)\sin(\frac{x}{2})}{\sin^2(\frac{x}{2})}.$$

As a consequence of the trigonometric identities above, we have

$$\sin\left((n+\frac{1}{2})x\right)\sin\left(\frac{x}{2}\right) = \frac{1}{2}\left(\cos(nx) - \cos((n+1)x)\right),$$

$$2\sin^2\left(\frac{x}{2}\right) = 1 - \cos x.$$