Math 511, Spring 2018
Assignment 3, due Wednesday, February 7

Problems to hand in:

1. Let $x > 0$. Prove that $2^{2x-1}B(x,x) = B(x, \frac{1}{2})$ by using (98) in the text. From this conclude *Legendre's duplication formula* (this is (102) in the text after a change of notation)
$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right).$$
Hint: Use that $2\sin\theta\cos\theta = \sin(2\theta)$ and that $\sin\theta$ is symmetric under reflections in $\theta = \frac{1}{2}$.

2. Prove that Γ is differentiable on $(0, \infty)$ with
$$\Gamma'(x) = \int_0^\infty e^{-t}t^{x-1}\log t\,dt.$$
Note: Since the domain of integration defining the gamma function is an unbounded interval, Theorem 7.16 isn't sufficient. Try the dominated convergence theorem in Exercise 12, Chapter 7 of Rudin.

3. Rudin, Chapter 9, #8. Prove this without appealing to Theorem 9.17. In other words, show directly that $f'(x)$ as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ satisfies $f'(x)u = 0$ for every $u \in \mathbb{R}^n$.

4. (Math 510 review) Let $I \subset \mathbb{R}$ be an interval with $x_0 \in I$ and let $f : I \to \mathbb{R}$. Prove or disprove the following statement:

   If f is differentiable on $I \setminus \{x_0\}$ and $\lim_{x\to x_0} f'(x)$ does not exist, then $f'(x_0)$ does not exist.

On your own: Rudin, Chapter 9, exercises 6, 7, 9, 10 and the following problems:

1. Show that the improper integral defining the gamma function $\int_0^\infty t^{x-1}e^{-t}dt$ is convergent for $x > 0$.

2. (a) Use Stirling's formula to prove that for any $c \in \mathbb{R}$,
$$\lim_{x\to\infty}\frac{\Gamma(x+c)}{x^c\Gamma(x)} = 1.$$
   (b) Use the beta function to prove that
$$\lim_{n\to\infty}\sqrt{n}\int_{-1}^{1}(1-x^2)^n\,dx = \sqrt{\pi}$$
   Note: This arises in the proof of Theorem 7.26 in Rudin. Defining c_n as in (47) and (48), this exercise shows that $\lim_{n\to\infty}\frac{\sqrt{n}}{c_n} = \sqrt{\pi}$.

Reading: Rudin, finish Chapter 8, begin Chapter 9.