Math 511, Spring 2018
Assignment 4, due Wednesday, February 14

Hand in solutions to all parts of the following three problems:

1. A function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is said to be *homogeneous of degree* λ if $f(tx) = t^\lambda f(x)$ for $t > 0$. If f is the zero function, then it is homogeneous of any degree, so in what follows we assume f is *nontrivial*, that is, it is not the zero function (i.e. $f(x) \neq 0$ for at least one x). We will also restrict attention to $\lambda \in \mathbb{R}$. Homogeneous functions are determined by their values on the unit sphere in that $f(x) = |x|^\lambda f(\frac{x}{|x|})$ for every $x \neq 0$.

   In what follows, suppose f is a continuous, nontrivial, homogeneous function of degree λ.

   (a) Show that $\lim_{x\to 0} f(x) = 0$ if and only if $\lambda > 0$. In these cases, we will extend f to the origin by setting $f(0) = 0$, thus defining a continuous function on all of $\mathbb{R}^n$.

   Note: On your own you should be able to see that $\lim_{x\to 0} f(x)$ does not exist if $\lambda < 0$ and that when $\lambda = 0$, $\lim_{x\to 0} f(x)$ exists if and only if f is constant.

   (b) Show that if $0 < \lambda < 1$, then f is not differentiable at 0.

   (c) Show that if $1 < \lambda$ then f is differentiable at 0 and $f'(0) = 0$.

   (d) Suppose $\lambda = 1$. Show that f is differentiable at 0 if and only if f is a linear map.

   (e) Suppose f is differentiable on $\mathbb{R}^n \setminus \{0\}$ and that $x \neq 0$. Prove *Euler's lemma*:

   $$\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}(x) = \lambda f(x).$$

   Hint: Define $g : (0, \infty) \to \mathbb{R}$ by $g(t) = f(tx)$ and compute $g'(t)$ in two different ways.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

   $$f(x,y) = \begin{cases} \frac{x^3 y}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Show that for any unit vector $\mathbf{u}$, the directional derivative of f at $(0,0)$ in the direction of $\mathbf{u}$ satisfies $\nabla_{\mathbf{u}} f(0,0) = 0$.

   (b) Prove that f is not differentiable at $(0,0)$.

   Hint: Consider the composition of f with the path $\mathbf{p}(t) = (t, t^2)$.

3. (a) Let $E \subset \mathbb{R}$ be a connected set and suppose $f : E \to \mathbb{R}$ is continuous. Prove that f is injective if and only if f is either strictly increasing or strictly decreasing.

   (b) Rudin, Chapter 9, # 16. Use the result in (a) to solve this problem.

On your own: Rudin, Chapter 9, Exercises 11[1], 17, 19 and the following problems:

1. Let Ω be an open subset of $\mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^m$ be differentiable on Ω. Let $a, b \in \Omega$ be two points such that the line segment joining a, b lies entirely in Ω. Given any $u \in \mathbb{R}^m$, show that there exists c on this line segment such that

$$u \cdot (F(b) - F(a)) = u \cdot (F'(c)(b - a)) \ .$$

   Then use this to give a second proof of Theorem 9.19 in Rudin.

Reading: Rudin Chapter 9, and notes on the inverse function theorem.

[1] This is not quite as easy as it looks since the functions are not assumed to be *continuously* differentiable and hence Theorem 9.21 does not apply.