Math 511, Spring 2018
Assignment 5, due Wednesday, February 28

Hand in solutions to all parts of the following 3 problems:

1. Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$ be a $\mathcal{C}^2(\Omega)$ function. Recall that the Hessian of f at a is the symmetric matrix defined by

$$\nabla^2 f(a) = \begin{bmatrix} D^2_{11}f(a) & D^2_{12}f(a) & \cdots & D^2_{1n}f(a) \\ D^2_{21}f(a) & D^2_{22}f(a) & \cdots & D^2_{2n}f(a) \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ D^2_{n1}f(a) & D^2_{n2}f(a) & \cdots & D^2_{nn}f(a) \end{bmatrix}$$

Recall that a matrix A is said to be *positive definite* (respectively *positive semi-definite*) if for every $x \neq 0$, $(Ax) \cdot x > 0$ (resp. $(Ax) \cdot x \geq 0$).

   (a) Suppose that for some $a \in \Omega$, $\nabla f(a) = 0$ and that the Hessian matrix of f at a is positive definite. Use a Taylor expansion of f to prove that there exists $\delta > 0$ such that $N_\delta(a) \subset \Omega$ and that $f(a) < f(x)$ for all $x \in N_\delta(a) \setminus \{a\}$, that is, f has a strict local minimum at a.

   Hint: Use that if $c = \min\{(Ax) \cdot x : |x| = 1\}$, then $(Ax) \cdot x \geq c|x|^2$ for all x. Then show that if $\|A - B\| \leq \frac{c}{2}$, then $(Bx) \cdot x \geq \frac{c}{2}|x|^2$, so that B is positive definite as well.

   (b) Now suppose that in addition, Ω is convex. The function f is said to be *convex* if it $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$ for each $a, b \in \Omega$ and each $0 < \lambda < 1$.

   Prove that f is convex if and only if its Hessian matrix is positive semi-definite for each $a \in \Omega$.

   Hint: Start by showing that if $a, b \in \Omega$ and $g(t) := f(a + t(b-a))$, then g is a convex function of one variable.

2. Rudin, Chapter 9, # 24.

   Hint: The range of $\mathbf{f}$ is an ellipse, find the equation of it. Be sure to justify that the range of $\mathbf{f}$ is contained in the ellipse and that each point in the ellipse is the image of some (x, y).

3. Let $I \subset \mathbb{R}$ be an open interval containing 0.

   (a) Suppose $f : I \to \mathbb{R}$ is $n + k$ times continuously differentiable with $n \geq 1$ and $k \geq 0$. Use Taylor's formula with integral remainder (in 1 variable) to show that there exists a $\mathcal{C}^k(I)$ function $p(s)$ such that

$$f(s) = f(0) + f'(0)s + \frac{f''(0)}{2}s^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}s^{n-1} + s^n p(s)$$

(b) Now let $f, g : I \to \mathbb{R}$ be $\mathcal{C}^4(I)$ functions whose Taylor polynomials of degree 2 at 0 are identical, with $f''(0), g''(0) \neq 0$. Given the previous part, this means that there exist $p, q \in \mathcal{C}'(I)$ and $a, b \in \mathbb{R}$, $b \neq 0$, such that

$$f(s) = a + bs^2 + s^3 p(s) \quad \text{and} \quad g(t) = a + bt^2 + t^3 q(t).$$

Prove that there exists a change of variables $s = s(t)$ defined for t in a neighborhood of 0 with $s'(0) = 1$ so that $f(s(t)) = g(t)$.

Hint: Use the implicit function theorem, verifying that

$$s(1 + sp(s)b^{-1})^{1/2} = t(1 + tq(t)b^{-1})^{1/2}$$

defines s as a function of t.

Note: We used this result when we proved Stirling's formula for the asymptotics of the Gamma function to argue that there exists $\delta > 0$ such that

$$\int_{-\delta}^{\delta} \left[(1+u)e^{-u}\right]^x \, du = \int_{-u^{-1}(\delta)}^{u^{-1}(\delta)} e^{-\frac{xs^2}{2}} u'(s)\, ds$$

for some increasing change of variables $u = u(s)$ defined on an interval about the origin $[-u^{-1}(\delta), u^{-1}(\delta)]$ with $u'(0) = 1$. In particular, both $(1 + u)e^{-u}$ and $e^{-\frac{s^2}{2}}$ have identical quadratic Taylor polynomials about $u = 0$ and $s = 0$ respectively.

On your own: Rudin, Chapter 9, Exercises 15, 19, 20, 21, 22, 23, 26, 27, and the following problems:

1. When defining differentiability for scalar functions f of one variable in Math 510, we allowed the domain of f to be a closed interval (see Definition 5.1 in Rudin). However, in defining differentiability for functions of several variables, we took the domain of f to be an open set in $\mathbb{R}^n$ (see Definition 9.11 in Rudin). Why do you think a closed interval can be taken in the single variable case, while f needs to be defined on an open set in the several variable case?

2. Derive the implicit function theorem as a consequence of the rank theorem.

   Note: As remarked in class, in the setting of the rank theorem from the typed notes, if $f : \Omega \subset \mathbb{R}^{n+m} \to \mathbb{R}^n$ has full rank n, then ψ can be taken to be the identity. You may need to appeal to a few elements of the proof of the rank theorem as well.

3. Returning to the definition of convex functions in Exercise 1b, show that f is convex if and only if the set $\{(x, y) \in \Omega \times \mathbb{R} : y \geq f(x)\}$ is a convex set in $\mathbb{R}^{n+1}$.

Reading: Finish Chapter 9 of Rudin, and the notes on Taylor's formula. Get ready for Wade's treatment of the Riemann integral in $\mathbb{R}^n$ starting at the end of Week 6 or the beginning of Week 7.