Math 511, Spring 2018
Assignment 6, Due Wednesday, March 7

Exercises to hand in:

1. Let E be a Jordan region in $\mathbb{R}^n$.
   (a) Prove that E^o and $\overline{E}$ are Jordan regions.
   (b) Prove that $\text{Vol}(E^o) = \text{Vol}(\overline{E}) = \text{Vol}(E)$.
2. Let E_1, E_2 be Jordan regions in $\mathbb{R}^n$.
   (a) Prove that if $E_1 \subset E_2$, then $\text{Vol}(E_1) \le \text{Vol}(E_2)$.
   (b) Prove that $E_1 \cap E_2$ and $E_1 \setminus E_2$ are Jordan regions.
   (c) Prove that if E_1, E_2 are nonoverlapping (that is, $E_1 \cap E_2$ is of volume zero), then
   $$\text{Vol}(E_1 \cup E_2) = \text{Vol}(E_1) + \text{Vol}(E_2).$$
3. Suppose $E \subset \mathbb{R}^n$ is compact and $f : E \to \mathbb{R}$ is continuous on E. Prove that the graph of f
   $$\big\{(x, f(x)) \in \mathbb{R}^{n+1} : x \in E\big\}$$
   is of volume zero in $\mathbb{R}^{n+1}$.

On your own: Wade 12.1.2, 12.1.7, 12.1.9 and the following 3 problems:

1. Find a set $E \subset \mathbb{R}^n$ such that $\overline{E^\circ} \neq \overline{E}^\circ$ that is, the closure of the interior of E is not equal to the the interior of the closure of E.

   Note: In arguments concerning outer sums (as well as others in this unit) it is sometimes tempting to use that $\overline{E^\circ} = \overline{E}^\circ$. This exercise is significant in that it shows that you *cannot* reason this way.
2. Use exercise above to prove the following for Jordan regions E, E_1, E_2
   (a) If $E_1 \subset E_2$, prove that
   $$\text{Vol}(E_1 \setminus E_2) = \text{Vol}(E_1) - \text{Vol}(E_2).$$
   (b) Prove that
   $$\text{Vol}(E_1 \cup E_2) = \text{Vol}(E_1) + \text{Vol}(E_2) - \text{Vol}(E_1 \cap E_2)$$
   (c) Prove that $\text{Vol}(E) > 0$ if and only if $E^o \neq \emptyset$.
3. A set $E \subset \mathbb{R}^n$ is said to be of *measure zero* if and only if given $\varepsilon > 0$, there is a sequence of rectangles $R_1, R_2, \ldots$ which cover E such that $\sum_{k=1}^{\infty} |R_k| < \varepsilon$.

(a) Prove that if $E \subset \mathbb{R}^n$ is of volume zero, then E is of measure zero.

(b) Prove that if $E \subset \mathbb{R}^n$ is countable, then E is of measure zero.

(c) Prove that there is a set $E \subset \mathbb{R}^2$ of measure zero which does not have zero area and, in fact, is not even a Jordan region.

Reading in Wade: §12.1, §12.2, §12.3