Math 511, Spring 2018
Assignment 7, Due Wednesday, March 21

Exercises to hand in:

1. (Wade 12.2.3) Let $E \subset \mathbb{R}^n$ be an open Jordan region and $x_0 \in E$. If $f : E \to \mathbb{R}$ is integrable on E and continuous at x_0, prove that

$$\lim_{r \to 0+} \int_{B_r(x_0)} f\, dV = f(x_0).$$

   Notes: Balls are Jordan regions as a consequence of Exercise 3 in Assignment 6 ($\partial B_r(x_0)$ is the sphere of radius r, which is the union of two graphs in $n-1$ variables). The exercises 12.2.9 and 12.4.6 in the "On your own" portion of this exercises can be seen to be corollaries of this one.

2. (Wade 12.2.10a) Suppose E is a Jordan region and that $f : E \to \mathbb{R}$ is integrable on E. If $f(E) \subseteq H$, for some compact set H and $\phi : H \to \mathbb{R}$ is continuous, prove that $\phi \circ f$ is integrable on E.

   Wade's hint, corrected: Let $\varepsilon > 0$ and choose $\delta > 0$ by uniform continuity of ϕ. Choose a grid $\mathcal{G}$ such that $U(f, \mathcal{G}) - L(f, \mathcal{G}) < \delta^2$. Then break $U(\phi \circ f, \mathcal{G}) - L(\phi \circ f, \mathcal{G})$ into two pieces: those j which satisfy $M_j(f) - m_j(f) < \delta$ and those j which satisfy $M_j(f) - m_j(f) \geq \delta$. These two pieces are small for different reasons.

3. Suppose E is a Jordan region and that $f, g : E \to \mathbb{R}$ are bounded functions. Suppose further that there exists a constant $K \geq 0$ such that

$$|f(x) - f(y)| \leq K|g(x) - g(y)|$$

   for all $x, y \in E$. Prove that if g is integrable on E, then so is f.

   Note: Be careful here, remember that "integrable" means that after extending the function to $\mathbb{R}^n$, then the upper and lower integrals over some rectangle $R \supset E$ are equal. So the fact that $|f(x) - f(y)| \leq K|g(x) - g(y)|$ for $x, y \in E$ does not mean the inequality persists for their extensions to $\mathbb{R}^n$ (on your own, find an example which shows this!).

On your own: Exercises 12.2.9, 12.3.8, and 12.4.6 in Wade and the following:

1. (Wade 12.2.5) Suppose $E_0, E \subset \mathbb{R}^n$ are Jordan regions with $E_0 \subset E$. If $f : E \to \mathbb{R}$ is integrable on E, show that f is also integrable on E_0.

2. Suppose f, g are integrable functions on a Jordan region E. Using prior exercises, show that f^2, $|f|$, fg, $\max(f, g)$, $\min(f, g)$ all define integrable functions on E.

3. Verify the details in Remark 12.16.

4. Let $\phi, \psi : [a,b] \to \mathbb{R}$ be bounded functions. Verify the following assertions at the top of p. 477 concerning 1 dimensional integrals, the last two holding for any partition P of $[a,b]$:

$$(U)\int_a^b (\phi(x)+\psi(x))\,dx \le (U)\int_a^b \phi(x)\,dx + (U)\int_a^b \psi(x)dx,$$

$$(L)\int_a^b (\phi(x)+\psi(x))\,dx \ge (L)\int_a^b \phi(x)\,dx + (L)\int_a^b \psi(x)dx,$$

$$(U)\int_a^b \phi(x)\,dx = \sum_{j=1}^n (U)\int_{x_{j-1}}^{x_j} \phi(x)\,dx,$$

$$(L)\int_a^b \phi(x)\,dx = \sum_{j=1}^n (L)\int_{x_{j-1}}^{x_j} \phi(x)\,dx.$$

Reading: Wade, §12.3, §12.4, §12.4. Review double and triple integrals as in Wade §12.3 working through computational exercises such as 12.3.3 and 12.3.4 as needed. They will reappear in our study of differential forms.