Math 511, Spring 2018
Assignment 8, Due Wednesday, March 28

Exercises to hand in:

1. (Wade 12.4.9) Let $\mathbf{v}_j = (v_{j1}, \dots, v_{jn}) \in \mathbb{R}^n$, $j = 1, \dots, n$ be a fixed set of n vectors in $\mathbb{R}^n$. The parallelepiped determined by $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is the set

$$\mathcal{P}(\mathbf{v}_1, \dots, \mathbf{v}_n) = \{t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n : t_j \in [0, 1]\},$$

and the determinant of the $\mathbf{v}_j$'s is the number

$$\det(\mathbf{v}_1, \dots, \mathbf{v}_n) := \det[v_{jk}].$$

Prove that

$$\mathrm{Vol}(\mathcal{P}(\mathbf{v}_1, \dots, \mathbf{v}_n)) = \det(\mathbf{v}_1, \dots, \mathbf{v}_n).$$

Note: This is rather direct from the change of variables formula when $\det(\mathbf{v}_1, \dots, \mathbf{v}_n) \neq 0$. But be fully rigorous about what happens when the determinant vanishes: argue that $\mathcal{P}$ is contained in a subspace of dimension $k < n$ and hence must have volume zero.

2. A mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an *affine transformation* if it is defined by $\phi(x) = Ax + b$, where A is a non-singular $n \times n$ matrix, Ax denotes matrix vector multiplication, and $b \in \mathbb{R}^n$. Suppose $E \subset \mathbb{R}^n$ is a Jordan region and that ϕ is an affine transformation.

   (a) Show that $\mathrm{Vol}(\phi(E)) = |\det A| \times \mathrm{Vol}(E)$.

   (b) The *centroid* of E is defined as the point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ where

$$\bar{x}_i = \frac{1}{\mathrm{Vol}(E)} \int_E x_i \, dV$$

   where the integral on the right is to be interpreted as the integral of the function $g(x) = x_i$ over the region E. Show that $\phi(\bar{x})$ is the centroid of $\phi(E)$.

3. (Polar coordinates on $\mathbb{R}^n$, $n \geq 2$) Let $\Phi_n : [0, \infty) \times [0, \pi]^{n-2} \times [0, 2\pi] \to \mathbb{R}^n$ be defined as $\Phi_n(\rho, \varphi_1, \dots, \varphi_{n-1}, \theta) = (x_1, \dots, x_n)$ with

$$x_1 = \rho \cos \varphi_1, \quad x_{n-1} = \rho \cos \theta \prod_{j=1}^{n-2} \sin \varphi_j, \quad x_n = \rho \sin \theta \prod_{j=1}^{n-2} \sin \varphi_j,$$

$$x_k = \rho \cos \varphi_k \prod_{j=1}^{k-1} \sin \varphi_j, \qquad k = 2, \dots, n-2$$

(cf. Theorem 12.69 in Wade). When $n = 2$, define $\Phi_n : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^2$ by $\Phi_n(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta)$

(a) For each $\rho > 0$, let $\mathbb{S}_\rho = \{x \in \mathbb{R}^n : |x|^2 = \rho^2\}$ denote the sphere of radius ρ about the origin and define $\Psi_{\rho,n} : [0,\pi]^{n-2} \times [0, 2\pi] \to \mathbb{R}^n$ by $\Psi_{\rho,n}(\varphi_1, \dots, \varphi_{n-2}, \theta) = \Phi_n(\rho, \varphi_1, \dots, \varphi_{n-1}, \theta)$. Show that the range of $\Psi_{\rho,n}$ is equal to $\mathbb{S}_\rho$ and that the restriction of $\Psi_{\rho,n}$ to $(0,\pi)^{n-2} \times [0, 2\pi)$ is one-to-one. Conclude that Φ_n is onto and that the restriction of Φ_n to $(0,\infty) \times (0,\pi)^{n-2} \times [0, 2\pi)$ is one-to-one.

Hint: Induction on n is effective here. Take $n = 2$ to be the base case, and this will follow from the treatment of the trigonometric functions in Chapter 8 of Rudin (i.e. Theorem 8.7).

(b) Let $\Delta_n(\rho, \varphi_1, \dots, \varphi_{n-1}, \theta) = |\det \Phi_n'(\rho, \varphi_1, \dots, \varphi_{n-1}, \theta)|$. Prove that

$$\begin{aligned}\Delta_n &= \rho^{n-1} \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin^2\varphi_{n-3} \sin\varphi_{n-2} \\ &= \rho^{n-1} \prod_{j=1}^{n-2} \sin^{n-1-j}\varphi_j.\end{aligned}$$

Hint: Again, induction on n is effective here. Use properties of determinants to show that $\Delta_n = \rho \sin^n \varphi_1 \Delta_{n-1} + \rho \cos^2 \varphi_1 \sin^{n-2} \varphi_1 \Delta_{n-1}$.

4. Let B_R denote the ball of radius $R > 0$ about the origin in $\mathbb{R}^2$ and let Φ_2 be as in the previous exercise. Suppose $f : \bar{B}_R \to \mathbb{R}$ is a function such that both f and $f \circ \Phi_2$ are integrable. Consider the validity of the polar coordinates formula

$$\int_{\bar{B}_R} f(x,y)\, dV = \int_{[0,R]\times[0,2\pi]} f(\rho\cos\theta, \rho\sin\theta)\, \rho\, dV.$$

(a) Explain why by itself, Theorem 12.46 in Wade does not imply this polar coordinates formula, even if just barely.

(b) Prove that this formula is valid anyway by appealing to Theorems 12.65 and 12.24.

(c) Prove that this formula is valid anyway by observing that Theorem 12.46 does apply to the regions Γ_ϵ defined below, then considering the limit $\epsilon \to 0+$:

$$\Gamma_\epsilon := \{(\rho, \theta) : 0 \le \theta \le 2\pi - \epsilon, \epsilon \le \rho \le R\}, \quad \epsilon > 0$$

Note: Taking the closed rectangle in right hand side of the formula is significant so that Fubini's theorem gives $\int_0^{2\pi}\int_0^R f(\rho\cos\theta, \rho\sin\theta)\rho d\rho d\theta$, which is better for computations. You don't need to address this here, but balls can be replaced by wedges $\{(\rho,\theta) : 0 \le \rho \le R, \theta_1 \le \theta \le \theta_2\}$ provided $0 < \theta_2 - \theta_1 \le 2\pi$. On your own, work out similar considerations for polar coordinates when $n \ge 3$.

On your own: Exercises 12.4.10, 12.4.11, 12.5.1, 12.5.2, 12.5.3, 12.5.4, and 12.5.5 in Wade.
Reading: Wade, §12.5, §12.6.