FOURIER TRANSFORMS OF SURFACE MEASURE ON THE SPHERE
MATH 565, FALL 2017

1. Fourier transform of surface measure on the sphere

Recall that the distribution \( u \) on \( \mathbb{R}^n \) defined as

\[
\langle u, \psi \rangle = \int_{\mathbb{S}^{n-1}} \psi(x) \, d\sigma(x)
\]

is compactly supported and hence defines a tempered distribution on \( \mathbb{R}^n \). Here \( d\sigma \) denotes the usual surface measure on \( \mathbb{S}^{n-1} \). Indeed, it is apparent that if \( \text{supp}(\psi) \cap \mathbb{S}^{n-1} = \emptyset \), then \( \langle u, \psi \rangle = 0 \).

As a consequence of our theorem on Fourier transforms of compactly supported distributions, we have that the distribution \( \hat{u} \) is given by integration against the \( C^\infty \) slowly increasing function

\[
(1.1) \quad \hat{u}(\xi) = \int_{\mathbb{S}^{n-1}} e^{-ix \cdot \xi} \, d\sigma(x).
\]

**Theorem 1.1.** Suppose \( \hat{u}(\xi) \) is as in (1.1). Then for \( |\xi| \geq 1 \),

\[
(1.2) \quad \hat{u}(\xi) = 2(2\pi)^{\frac{n-1}{2}} |\xi|^{-\frac{n-1}{2}} \cos \left( |\xi| - \frac{\pi(n-1)}{4} \right) + R(\xi)
\]

where \( R(\xi) = O(|\xi|^{\frac{2}{n+1}}) \) as \( |\xi| \to \infty \), that is, \( |R(\xi)| \leq C|\xi|^{-\frac{n+1}{2}} \) for some constant \( C \).

**Proof.** We begin by observing that is \( \hat{u}(\xi) \) is radial. Indeed, given any orthogonal matrix \( A \),

\[
\hat{u}(A\xi) = \int_{\mathbb{S}^{n-1}} e^{-ix \cdot (A\xi)} \, d\sigma(x) = \int_{\mathbb{S}^{n-1}} e^{-ix^T A^T \xi} \, d\sigma(x) = \int_{\mathbb{S}^{n-1}} e^{-iz \cdot \xi} \, d\sigma(z) = \hat{u}(\xi)
\]

where in the second to last identity we used that \( d\sigma \) is invariant under the change of variables \( z = A^T x \). Thus in what follows, we may assume that \( \xi = \lambda e_n \), where \( e_n = (0, \ldots, 0, 1) \) is the \( n \)-th standard basis vector. We stress that this means \( |\xi| = \lambda \). We are then left to compute

\[
(1.3) \quad \hat{u}(\lambda e_n) = \int_{\mathbb{S}^{n-1}} e^{-i\lambda x_n} \, d\sigma(x).
\]

We now parameterize \( \mathbb{S}^{n-1} \) by letting \( y \) denote variables in \( \mathbb{R}^{n-1} \) and viewing the upper and lower hemispheres as a graph \( y \mapsto (y, \pm \sqrt{1 - |y|^2}) \) over the unit ball \( B(0, 1) \subset \mathbb{R}^{n-1} \). This yields

\[
(1.4) \quad \hat{u}(\lambda e_n) = \sum_{\pm} \int_{B(0,1)} e^{\mp i\lambda \sqrt{1-|y|^2}} (1 - |y|^2)^{-1/2} \, dy.
\]

Here we have used that

\[
(1.5) \quad \nabla \left( \sqrt{1 - |y|^2} \right) = \frac{-y}{\sqrt{1 - |y|^2}}, \text{ so that } \sqrt{1 + |\nabla(\sqrt{1 - |y|^2})|^2} = (1 - |y|^2)^{-1/2}.
\]

Hence this last expression appears in (1.4) on account of the usual formula for the surface measure element determined by a graph in \( \mathbb{R}^n \). The first expression here shows that the only critical points of the oscillatory integral defining \( \hat{u} \) occur at \( y = 0 \) which in turn corresponds to the north and south poles. Note that moreover, (1.4) implies that

\[
\hat{u}(\lambda e_n) = 2\text{Re} \left( \int_{B(0,1)} e^{i\lambda \sqrt{1-|y|^2}} \frac{1}{\sqrt{1 - |y|^2}} \, dy \right),
\]
showing that \( \hat{u}(\xi) \) is always real valued. Note that this is almost the real part of an “I(\lambda)” form that we typically like to write, the only difference being that the amplitude functions are not \( C^\infty \) here. This does not turn out to be of much significance since away from a neighborhood of the origin, we have a phase function with no critical points.

We are now led to define

(1.6) \[ I(\lambda) = \int e^{i\lambda \phi(y)} \psi(y)(1 - |y|^2)^{-1/2} dy \]

where \( \psi \in C_c^\infty(B(0,1/2)) \) satisfies \( \phi = 1 \) on \( B(0,1/4) \) and \( \phi(y) = \sqrt{1 - |y|^2} \). We then have

(1.7) \[ \hat{u}(\lambda e_n) = 2\text{Re}(I(\lambda)) + \sum_{\pm} \int_{B(0,1)} e^{\mp i\lambda \sqrt{1-|y|^2}} (1 - \psi)(y)(1 - |y|^2)^{-1/2} dy. \]

The second term here turns out to be \( \mathcal{O}(\lambda^{-N}) \) for any \( N \) since the phase function in the oscillatory integral has no critical points. To be fully rigorous about this, we should reparameterize this as a surface integral in a tubular neighborhood of the equator on the sphere, e.g. using spherical coordinates, but we will wave our hands over this technicality.

We are now reduced to computing the asymptotics of \( I(\lambda) \) using the stationary phase formula. Again (1.5) shows that the only critical point for \( I(\lambda) \) is at the origin. Moreover,

\[ \nabla \phi(0) = -\frac{I_{n-1}}{\sqrt{1 - |y|^2}} \bigg|_{y=0} + \frac{yy^T}{(\sqrt{1 - |y|^2})^3} \bigg|_{y=0} = -I_{n-1} \]

where \( I_{n-1} \) is the identity matrix in \( n-1 \) variables. Hence the phase is nondegenerate at the critical points. The stationary phase formula now gives us that

\[ I(\lambda) = (2\pi)^{\frac{n-1}{2}} \lambda^{-\frac{n+1}{2}} \left| \det \nabla^2 \phi(0) \right|^{-\frac{1}{4}} e^{\frac{\lambda}{4} \text{sgn} \nabla^2 \phi(0)} e^{i\lambda \phi(0)} \frac{\psi(0)}{\sqrt{1 - 0^2}} + \mathcal{O}(\lambda^{-\frac{n+1}{2}}). \]

The very last factor in the first term is clearly 1, but we have included it for emphasis as it is the result of evaluating the amplitude at \( y = 0 \). Since \( |\det \nabla^2 \phi(0)| = 1 \), \( \text{sgn} \nabla^2 \phi(0) = -(n-1) \), and \( \phi(0) = 1 \), we now have

\[ I(\lambda) = (2\pi)^{\frac{n-1}{2}} \lambda^{-\frac{n+1}{2}} e^{i\lambda^{1\frac{1}{4}}(n-1)} + \mathcal{O}(\lambda^{-\frac{n+1}{2}}). \]

Taking twice the real part of this expression and recalling that \( |\xi| = \lambda \) then results in (1.2). \( \square \)

**Corollary 1.2.** The function \( \hat{u}(\xi) \) in (1.1) satisfies

\[ \hat{u}(\xi) = \Re \left( e^{i|\xi|b(|\xi|)} \right) \]

where \( b : [0, \infty) \to \mathbb{C} \) is a smooth function satisfying for \( j \geq 0 \)

(1.8) \[ \left| \frac{d^j}{d\lambda^j} b(\lambda) \right| \leq C_j (1 + \lambda)^{-\frac{n-1}{2} - j}. \]

**Proof.** First observe that since \( \hat{u} \) is \( C^\infty \) and radial, then when \( \lambda = |\xi| \leq 1 \), (1.8) follows since each derivative is bounded on the closed unit ball about the origin. Hence we only need to treat the case \( \lambda = |\xi| > 1 \), which is assumed in the following.

In (1.6), (1.7), we used that \( \hat{u} \) is radial to see that with \( |\xi| = \lambda \),

\[ \hat{u}(\xi) = 2\text{Re}(I(\lambda)) + E(\lambda), \quad E(\lambda) = \sum_{\pm} \int_{B(0,1)} e^{\mp i\lambda \sqrt{1-|y|^2}} (1 - \psi)(y)(1 - |y|^2)^{-1/2} dy \]

so that \( E(\lambda) \) is the contribution of the second term in (1.7). As before, the \( E(\lambda) \) contribution can be reparameterized so that it is an oscillatory integral with no critical points, and hence satisfies
stronger bounds

\[ (1.9) \quad \left| \frac{d^j}{d\lambda^j} E(\lambda) \right| \leq C_{j,N} \lambda^{-N} \]

for any \( N \geq 0 \). Since \( E(\lambda) \) is real valued,

\[ \tilde{u}(\xi) = 2\text{Re}(I(\lambda) + E(\lambda)) = \text{Re}(2I(\lambda) + E(\lambda)). \]

We therefore take

\[ b(\lambda) = 2e^{-i\lambda}I(\lambda) + e^{-i\lambda}E(\lambda). \]

Given (1.9), and the product rule, the second term here satisfies the bounds in (1.8). Exercise 3b in Assignment 5 then yields the desired bounds on \( \frac{d^j}{d\lambda^j}(2e^{-i\lambda}I(\lambda)) \).

\[ \square \]

**Theorem 1.3.** Let \( B = B(0,1) \subset \mathbb{R}^n \) be the unit ball in \( \mathbb{R}^n \). There is a constant \( C \) such that

\[ (1.10) \quad |\hat{1}_B(\xi)| \leq C(1 + |\xi|)^{-\frac{n+1}{2}} \quad \text{for all } \xi \in \mathbb{R}^n. \]

**Proof.** The case \( n = 1 \) is straightforward and left to the reader, so assume \( n \geq 2 \) in what follows. Since \( 1_B \in L^1(\mathbb{R}^n) \), its Fourier transform is bounded, so the bound (1.10) is satisfied when \( |\xi| \leq 1 \).

It thus suffices to show that \( |\hat{1}_B(\xi)| \leq C|\xi|^{-\frac{n+1}{2}} \) for \( |\xi| > 1 \).

In what follows \( C \) is a constant that may change from line to line. Observe that

\[ \hat{1}_B(\xi) = \int_B e^{-i\xi \cdot y} dy = \int_0^1 \left( \int_{S^{n-1}} e^{-i\xi \cdot r \xi} d\sigma(x) \right) r^{n-1} dr \]

where we have used polar coordinates \( y = r \xi \) with \( x \in S^{n-1} \). The integral in parentheses is \( \tilde{u}(r\xi) \) where \( \tilde{u} \) is defined in (1.1). Corollary 1.2 then shows that \( \hat{1}_B(\xi) \) is the real part of the following integral

\[ \int_0^1 e^{ir|\xi|} b(r|\xi|) r^{n-1} dr = \frac{1}{i |\xi|} \int_0^1 \partial_r \left( e^{ir|\xi|} \right) b(r|\xi|) r^{n-1} dr \]

\[ = \frac{1}{i |\xi|} e^{ir|\xi|} b(r|\xi|) r^{n-1} \bigg|_{r=0}^{r=1} - \frac{1}{i |\xi|} \int_0^1 e^{ir|\xi|} \partial_r \left( b(r|\xi|) r^{n-1} \right) dr \]

\[ = \frac{e^{i|\xi|} b(|\xi|)}{i |\xi|} - \frac{1}{i |\xi|} \int_0^1 e^{ir|\xi|} \left( b'(r|\xi|) |\xi| r^{n-1} + (n-1)b(r|\xi|) r^{n-2} \right) dr. \]

Here we have used that \( \frac{1}{i |\xi|} e^{ir|\xi|} b(r|\xi|) r^{n-1} \big|_{r=0} = 0 \). The first term here is bounded using (1.8):

\[ \left| \frac{e^{i|\xi|} b(|\xi|)}{i |\xi|} \right| = |\xi|^{-1} |b(|\xi|)| \leq C|\xi|^{-1} |\xi|^{-\frac{n+1}{2}} = C|\xi|^{-\frac{n+1}{2}} \]

We then have

\[ \left| \frac{1}{i |\xi|} \int_0^1 e^{ir|\xi|} \left( b'(r|\xi|) |\xi| r^{n-1} + (n-1)b(r|\xi|) r^{n-2} \right) dr \right| \]

\[ \leq \frac{1}{|\xi|} \int_0^1 |b'(r|\xi|) |\xi| r^{n-1} + (n-1)b(r|\xi|) r^{n-2} dr \]

\[ \leq C \frac{1}{|\xi|} \int_0^1 (r|\xi|)^{-\frac{n+1}{2} - 1} |\xi| r^{n-1} + (r|\xi|)^{-\frac{n+1}{2} - 2} r^{n-2} dr = \frac{2C}{|\xi|^{\frac{n+1}{2}}} \int_0^1 r^{\frac{n-3}{2}} dr, \]

where we have used (1.8) in the second inequality and that \( (1 + r|\xi|)^{-1} \leq (r|\xi|)^{-1} \). The integral on the very right here is always finite since our assumption \( n \geq 2 \) means that \( (n-3)/2 > -1 \). Hence the second term in (1.11) is bounded by \( C|\xi|^{-\frac{n+1}{2}} \) as well. \( \square \)
Theorem 1.4. Suppose $n \geq 2$ and that $N(\lambda) = \# \{ m \in \mathbb{Z}^n : |m| = \lambda \}$ is the number of integer lattice points in the ball of radius $\lambda$ about the origin. Then

$$N(\lambda) = Vol(B(0))\lambda^n + R(\lambda), \text{ where } |R(\lambda)| \leq C\lambda^{n-2+\frac{2}{n+1}}.$$  
(1.12)

In other words, $N(\lambda) = Vol(B(0))\lambda^n + O(\lambda^{n-2+\frac{2}{n+1}})$.

Note that since $n - 2 + \frac{2}{n+1} < n - 1 < n$, the remainder in (1.12) is indeed of lower order.

Proof. Recall the Poisson summation formula

$$\sum_{m \in \mathbb{Z}^n} g(2\pi m) = (2\pi)^{-n} \sum_{m \in \mathbb{Z}^n} \hat{g}(m).$$

We will instead use the following rescaled version of this identity:

$$\sum_{m \in \mathbb{Z}^n} f(m) = \sum_{m \in \mathbb{Z}^n} \hat{f}(2\pi m).$$

Indeed, taking $g(x) = f(x/2\pi)$, we have that $\hat{g}(\xi) = (2\pi)^n \hat{f}(2\pi \xi)$, so (1.13) does indeed follow from the first version of the Poisson summation formula.

Let $\chi_\eta = 1_{B(0, \eta)}$ denote the characteristic function of the ball of radius $\eta$. Note that

$$N(\lambda) = \sum_{m \in \mathbb{Z}^n} \chi_\lambda(m).$$

Hence ideally, we would apply the Poisson summation formula (1.13) with $f = \chi_\eta$ and $\eta = \lambda$, but we do not have enough decay of $\chi_\eta$ to satisfy the hypotheses of the theorem, which require that $f, \hat{f}$ are $O(|\xi|^{-n-\varepsilon})$ as $|\xi| \to \infty$. Indeed, $\chi_\eta(x) = \chi_1(x/\eta)$ (since $\chi_1$ is just the characteristic function of the unit ball) and hence

$$|\tilde{\chi}_\eta(\xi)| = \eta^n |\tilde{\chi}_1(\eta\xi)| \leq C\eta^n (1 + \eta |\xi|)^{-\frac{n+1}{2}},$$

showing that the decay at infinity is insufficient to apply (1.13). We therefore have to regularize $\chi_\eta$, by convolving with an approximation to the identity.

Let $\beta \in C^\infty_c(B(0, 1))$ be a radial, nonnegative function satisfying $\int \beta(x) dx = 1$ and set $\beta_\varepsilon(x) = \varepsilon^{-n} \beta(x/\varepsilon)$. We then define $\tilde{\chi}_\eta = \chi_\eta * \beta_\varepsilon$. In what follows, we will treat $\varepsilon$ as fixed, but all the constants in the inequalities which follow can be taken independently of $\varepsilon$ as long as $0 < \varepsilon < 1$. In particular, we will use that for sufficiently large, $\eta \approx \eta \pm \varepsilon$.

As in the proof of the $C^\infty$ Urysohn lemma, we have that $0 \leq \tilde{\chi}_\eta(x) \leq 1$ for each $x$ and that

$$\tilde{\chi}_\eta(x) = \begin{cases} 1, & |x| \leq \eta - \varepsilon \\ 0, & |x| \geq \eta + \varepsilon. \end{cases}$$

Indeed, since $\text{supp}(\beta_\varepsilon(x - \cdot)) \subset \{ y : |y| < \varepsilon \}$,

$$\tilde{\chi}_\eta(x) = \int_{B(0, \eta)} \beta_\varepsilon(x - y) dx = \int_{B(0, \eta) \cap B(x, \varepsilon)} \beta_\varepsilon(x - y) dx,$$

so if $|x| \leq \eta - \varepsilon$, $B(0, \eta) \cap B(x, \varepsilon) = B(x, \varepsilon)$ so that the integral is 1, and if $|x| \geq \eta + \varepsilon$, $B(0, \eta)$ and $B(x, \varepsilon)$ are disjoint, so the integral vanishes. Moreover, for any $x$, we have $0 \leq \tilde{\chi}_\eta(x) \leq 1$ since $\beta \geq 0$ and $\int \beta_\varepsilon = 1$.

We are now interested in $\tilde{\chi}_{\lambda \pm \varepsilon}$. The above shows that

$$\text{supp}(\tilde{\chi}_{\lambda - \varepsilon}) \subset B(0, (\lambda - \varepsilon) + \varepsilon) = B(0, \lambda) = \{ x : \chi_\lambda(x) = 1 \},$$

$$\{ x : \tilde{\chi}_{\lambda + \varepsilon}(x) = 1 \} \supset B(0, (\lambda - \varepsilon) + \varepsilon) = B(0, \lambda) = \text{supp}(\chi_\lambda),$$

which implies the following pointwise inequalities for any $x \in \mathbb{R}^n$

$$\tilde{\chi}_{\lambda - \varepsilon}(x) \leq \chi_\lambda(x) \leq \tilde{\chi}_{\lambda + \varepsilon}(x).$$

(1.16)
Consequently, defining \( \tilde{N}(\eta) = \sum_{m \in \mathbb{Z}^n} \tilde{\chi}_\eta(m) \) as an approximation to (1.14), we have that
\[
(1.17) \quad \tilde{N}(\lambda - \varepsilon) \leq N(\lambda) \leq \tilde{N}(\lambda + \varepsilon).
\]
Indeed, this is the result of taking \( m \) in (1.16) and summing over all \( m \in \mathbb{Z}^n \).

The main idea is now to show that for \( \eta \) sufficiently large,
\[
(1.18) \quad \tilde{N}(\eta) = \text{Vol}(B)\eta^n + \tilde{R}(\eta), \quad \text{where} \quad |\tilde{R}(\eta)| \leq C\eta^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}.
\]
As soon as we show this, then (1.12) will follow. Indeed, since \( \lambda \pm \varepsilon \approx \lambda \), we have that
\[
(1.19) \quad \tilde{N}(\lambda \pm \varepsilon) = \text{Vol}(B)(\lambda \pm \varepsilon)^n + \tilde{R}(\eta) = \text{Vol}(B)\lambda^n + O(\lambda^{n-1}\varepsilon) + O(\lambda^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}).
\]
It is verified that taking \( \varepsilon = \lambda^{\frac{n-1}{2}} \) then ensures that the lower order terms are the same:
\[
\lambda^{n-1}\varepsilon = \lambda^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}} = \lambda^{n-2+\frac{2}{n+1}},
\]
Hence \( \tilde{N}(\lambda \pm \varepsilon) = \text{Vol}(B)\lambda^n + R_\pm(\lambda) \), where \( R_\pm(\lambda) = O(\lambda^{n-2+\frac{2}{n+1}}) \), which shows that
\[
R_-(\lambda) = \tilde{N}(\lambda - \varepsilon) - \text{Vol}(B)\lambda^n \leq N(\lambda) - \text{Vol}(B)\lambda^n \leq \tilde{N}(\lambda + \varepsilon) - \text{Vol}(B)\lambda^n = R_+(\lambda),
\]
that is, \( |N(\lambda) - \text{Vol}(B)\lambda^n| \leq \max(R_-(\lambda), R_+(\lambda)) \leq C\lambda^{n-2+\frac{2}{n+1}} \), concluding the proof of (1.12).

Before proceeding with (1.18), it is interesting to note that \( \varepsilon = \lambda^{\frac{n-1}{2}} \) is indeed the optimal choice given the two error terms in (1.19). Indeed, we ultimately want the remainder in theorem to be as small as possible, so we want to take \( \varepsilon \) as small as possible. But the second error term in (1.19) is an inverse power of \( \varepsilon \), penalizing us in taking \( \varepsilon \) small. The optimal choice thus results from taking \( \varepsilon \) to satisfy \( \lambda^{n-1}\varepsilon = \lambda^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}} \), which is the same as taking \( \varepsilon = \lambda^{\frac{n-1}{2}} \).

We are now left to show (1.18). Applying (1.13) we have that
\[
\tilde{N}(\eta) = \sum_{m \in \mathbb{Z}^n} \tilde{\chi}_\eta(m) = \sum_{m \in \mathbb{Z}^n} \tilde{\chi}_\eta(2\pi m) = \sum_{m \in \mathbb{Z}^n} \tilde{\chi}_\eta(2\pi m) \hat{\beta}(2\pi \varepsilon m),
\]
using the usual formula for the Fourier transform of a convolution and that \( \hat{\beta}(\xi) = \hat{\beta}(\varepsilon \xi) \) using the usual dilation formula. Note that the \( m = 0 \) term in the sum here is
\[
\hat{\chi}_\eta(0) \hat{\beta}(0) = \text{Vol}(B)\eta^n.
\]
Indeed, the value of the Fourier transform at the origin is simply the mean of the function, so \( \hat{\chi}_\eta(0) = \int 1_{B(0,\eta)} = \text{Vol}(B)\eta^n \) and \( \hat{\beta}(0) = \int \beta = 1 \).

It remains to show that
\[
\left| \sum_{m \neq 0} \tilde{\chi}_\eta(2\pi m) \hat{\beta}(2\pi \varepsilon m) \right| \leq C\eta^{\frac{n-1}{2}}\varepsilon^{-\frac{n-1}{2}}.
\]
Treating the sum as a Riemann sum for the integral \( \int_{|\xi| \geq 1/2} \tilde{\chi}_\eta(2\pi \xi) \hat{\beta}(2\pi \varepsilon \xi) \), we have
\[
\left| \sum_{m \neq 0} \tilde{\chi}_\eta(2\pi m) \hat{\beta}(2\pi \varepsilon m) \right| \approx \left| \int_{|\xi| \geq 1/2} \tilde{\chi}_\eta(2\pi \xi) \hat{\beta}(2\pi \varepsilon \xi) d\xi \right|
\leq C\eta^n \int_{|\xi| \geq 1/2} (1 + |\eta|2\pi \xi)|^{-\frac{n+1}{2}}(1 + 2\pi \varepsilon \xi)^{-(n+1)} d\xi
\approx C\eta^n \int_{|\xi| \geq 1/2} (1 + |\eta|\xi)|^{-\frac{n+1}{2}}(1 + \varepsilon |\xi|)^{-(n+1)} d\xi
\]
where we have used (1.15) and that \( \hat{\beta} \in S(\mathbb{R}^n) \) and hence rapidly decreasing\(^1\). We now estimate the very last integral. When \( |\xi| \leq \varepsilon^{-1} \), the second factor in the integrand does not yield any appreciable

\(^1\)Getting rid of the \( 2\pi \) in the last expression isn’t really needed, it just makes the computations a little neater.
decay, so we will split the integral into regions where $1/2 \leq |\xi| \leq \varepsilon^{-1}$ and $|\xi| > \varepsilon^{-1}$. Taking polar coordinates in the integral we are reduced to estimating

$$
\eta^n \int_{r \geq 1/2} \frac{1}{\varepsilon} \frac{\eta r^{n+1}}{r^{n-1} dr} (1 + \varepsilon r)^{-(n+1)} r^{n-1} dr
\leq \eta^n \int_{1/2}^{\varepsilon^{-1}} (\eta r) \frac{n+1}{2} r^{n-1} dr + \eta^n \int_{\varepsilon^{-1}}^{\infty} (\eta r) \frac{n+1}{2} (\varepsilon r)^{-(n+1)} r^{n-1} dr
$$

The first integral can be estimated as

$$
\eta^n \int_{1/2}^{\varepsilon^{-1}} (\eta r) \frac{n+1}{2} r^{n-1} dr = \eta^n \frac{n+1}{2} \int_{1/2}^{\varepsilon^{-1}} \frac{r^{n-1}}{n-1} dr = \eta^n \frac{n+1}{2} \int_{r=1/2}^{r=\varepsilon^{-1}} \frac{r^{n-1}}{n-1} dr = \frac{2}{n-1} \eta^n \frac{n+1}{2} \varepsilon^{-\frac{n-1}{2}}.
$$

The second integral can be estimated as

$$
\eta^n \int_{\varepsilon^{-1}}^{\infty} (\eta r) \frac{n+1}{2} (\varepsilon r)^{-(n+1)} r^{n-1} dr = \eta^n \frac{n+1}{2} \frac{\varepsilon^{-(n+1)}}{n-1} \int_{\varepsilon^{-1}}^{\infty} \frac{r^{n-1}}{n-1} dr = \eta^n \frac{n+1}{2} \frac{\varepsilon^{-(n+1)}}{n-1} \int_{r=\varepsilon^{-1}}^{r=\infty} \frac{r^{n-1}}{n-1} dr = \eta^n \frac{n+1}{2} \frac{\varepsilon^{-(n+1)}}{n-1} \left[ \frac{2}{n+3} \varepsilon^{-\frac{n+3}{2}} \right]_{\varepsilon^{-1}}^{\infty}
$$

and this last expression is $O(\eta^n \varepsilon^{-(n+1)} \varepsilon^{\frac{n+3}{2}}) = O(\eta^n \varepsilon^{-\frac{n-1}{2}}).$  \hfill \Box