Artin-Hasse exp

(D) $\mu(1)=1,\ \mu(n)=0$ if $p^2 \mid n$, $\mu(p_1\cdots p_r)=(-1)^r$

(P) $\sum_{d\mid n}\mu(d) = \begin{cases} 1, & n=1 \\ 0, & n\neq 1\end{cases}$ $\quad$ [Pf. $\sum_{d\mid n}\mu(d) \underset{n=p_1^{e_1}p_2^{e_2}\cdots p_s^{e_s}}{=} \sum_t \mu(p_1^{i_1}\cdots p^{i_t}) = 1-\binom{t}{1}+\binom{t}{2}-\binom{t}{3}\cdots = (1-1)^t = 0$]

(P) $\exp(x) \overset{\text{in } \mathbb{Q}[[x]]}{=} \prod_{n=1}^{\infty}(1-x^n)^{-\mu(n)/n}$ $\quad$ [Pf. Let $f(x)=$ RHS. Then $\log f(x) = \sum_{n\geq 1}\frac{\mu(n)}{n}\sum_{k\geq 1}\frac{x^{nk}}{k}$

$$= \sum_{n\geq 1}\sum_{k\geq 1}\mu(n)\frac{x^{nk}}{nk} \underset{\substack{nk=m\\ d=n}}{=} \sum_m\Big(\sum_{d\mid m}\mu(d)\Big)\frac{x^m}{m} = x \ ]$$

(D) $AH(x) := \prod_{\substack{n\geq 1\\(p,n)=1}}(1-x^n)^{-\mu(n)/n}$ conv for $|x|_p<1$ $\quad$ [b/c $-\mu(n)/n\in\mathbb{Z}_p$ for $p\nmid n$]

(P) $AH(x) = \exp\left(x+\frac{x^p}{p}+\frac{x^{p^2}}{p^2}+\cdots\right)$ $\quad$ [in particular RHS $\in \mathbb{Z}_p[[x]]$]

Pf. $\log AH(x) = \sum_{\substack{n\geq 1\\ p\nmid n}}\frac{\mu(n)}{n}\sum_{k\geq 1}\frac{x^{nk}}{k} = \sum_m\Big(\sum_{\substack{d\mid m\\ p\nmid d}}\mu(d)\Big)\frac{x^m}{m} = \sum_{i\geq 0}\frac{x^{p^i}}{p^i}$ b/c:

(L) $\sum_{\substack{p\nmid d\\ d\mid m}}\mu(d) = \begin{cases}1 & \text{if } m=p^i\\ 0 & \text{if not.}\end{cases}$ $\quad$ [Pf. Say $m=p^e n$, $p\nmid n$, $n\neq 1$. Then LHS $=\sum_{d\mid n}\mu(d)=0$]

Picture of $\mathbb{Q}_p$