COMPLEX MULTIPLICATION

Euler, Gauss ($\ddagger$)

Kronecker, Weber, Deuring (T) $\forall$ ab xtn of $\mathbb{Q}$ emb into some $\mathbb{Q}(e(r))$. (R) Kron Jug traum: ($\ddagger$)

(D) A lattice $\Lambda \subset \mathbb{C}$ is a discr sgr w/ $\mathbb{R}$-span $= \mathbb{C}$; $\mathrm{End}(\Lambda) = \{\alpha \in \mathbb{C} \mid \alpha\Lambda \subset \Lambda\}$

(D) An ell curve/$\mathbb{C}$ is a $\mathbb{C}$-an grA $\simeq \mathbb{C}/$lattice; (P) $\mathrm{End}(\mathbb{C}/\Lambda) \simeq \mathrm{End}(\Lambda)$

(P) $\{$ell curves$\}/$iso $\overset{(*)}{\simeq} \mathbb{H}/SL_2(\mathbb{Z}) \overset{(**)}{\simeq} \mathbb{C}$

(D) An order in a quadr im field $K \subset \mathbb{C}$ is a subring $\mathcal{O} \subset K$ which is also a lattice. (○)

(P) $A = \mathbb{C}/\Lambda \Rightarrow \mathrm{End}(A)$ is either $\mathbb{Z}$ or an order $\mathcal{O}$ in an im quadr field.

(D) A has CM by $\mathcal{O}$ if $\mathrm{End}(A) = \mathcal{O}$. A has CM if it has CM by some $\mathcal{O}$

(P) $\{$ell curves w/ CM by $\mathcal{O}\}/$iso $\simeq \{$rank 1 f-g proj $\mathcal{O}$-mods$\}/$iso $=: \mathrm{Pic}(\mathcal{O})$

$\mathbb{C}/\Lambda \leftrightarrow \Lambda$; also RHS finite. Also $A = \mathbb{C}/\langle 1, \tau\rangle$ has CM $\Leftrightarrow$ τ quadr/$\mathbb{Q}$. (○○)

(T) $\tau \in \mathbb{H}$ quadratic over $\mathbb{Q}$ $\Rightarrow$ $j(\tau)$ an alg. integer and $H := \mathbb{Q}(\tau, j(\tau))$ is an abelian xtn of $K = \mathbb{Q}(\tau)$. If in add $\mathrm{End}(\langle 1,\tau\rangle) = \mathcal{O}_K$ then 1) any ab ext of K unr† over K is cont in H, 2) $\mathrm{Gal}(H/K) \simeq_{nat} \mathrm{Pic}(\mathcal{O}_K)$, 3) a pr id of $\mathcal{O}_K$ is princ $\Leftrightarrow$ it splits compl in $\mathcal{O}_H$ (○̸)

"Main Thm of CM"

(EG) $j(\sqrt{-14}) = 2^3\left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2}-1}\right)^3$ (Weber)

Weierstrass: ell curves as cubics

Eichler-Shimura-Wiles maps from $\mathbb{H}/\Gamma$ to ell curves/$\mathbb{Q}$

Heegner & BSD

Poonen-B

($\ddagger$) $e: \mathbb{C} \to \mathbb{C}$, $e(z) = e^{2\pi i z}$. $r \in \mathbb{Q} \Rightarrow e(r)$ alg integer & $\mathbb{Q}(e(r))$ abel xtn of $\mathbb{Q}$

($\ddagger$) Replace $\mathbb{Q}$ by # field K & find analogue of e for K.

($**$) j unique $SL_2(\mathbb{Z})$-inv holo fcn: $\mathbb{H} \to \mathbb{C}$ that starts w/ $\frac{1}{q} + \cdots$

($\dagger$) "unr" means all fin primes are unr & any real place can only be xtnded to real places (latter automatic for K which has no real emb)

(○̸) For K arb H called Hilb class field of K if 1) holds; 2) & 3) are conseq of 1)

(○○) Moreover A has CM by $\mathcal{O}$ in $K \Rightarrow K = \mathbb{Q}(\tau)$.

(○) $\exists!$ a unique maximal ord $\mathcal{O}_K$ & $\forall$ other ord is uniquely det by its index in $\mathcal{O}_K$, called its conductor

($*$) $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$ ↻ $SL_2(\mathbb{Z})$, $\mathbb{C}/\langle 1, \tau\rangle \leftrightarrow \tau$ | ($**$) Via $j: \mathbb{H} \to \mathbb{C}$, $j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots$, $q = \exp(2\pi i \tau)$