# DIFFERENTIAL ALGEBRA - A SCHEME THEORY APPROACH.

HENRI GILLET

Abstract. Two results in Differential Algebra, Kolchin's irreducibility theorem, and a result on descent of projective varieties (due to Buium) are proved using methods of "modern" or "Grothendieck style" algebraic geometry.

## 0. Introduction

The goal of this paper is to approach some results in differential algebra from the perspective, and using the results of, modern algebraic geometry and commutative algebra. In particular we shall see new proofs of two results: Kolchin's Irreducibility Theorem, and Buium's result describing the minimal field of definition of a projective variety over an algebraically closed field of characteristic zero.

The first section of the paper describes the construction of prolongations (which associate to a an algebraic variety $X$ over a differential field the ring of differential polynomial functions on $X$) from the point of view of adjoint functors. This allows us to give simple proofs of several properties of the "prolongation" operation, especially how the functor behaves with respect to formally smooth and formally étale morphisms. In particular we observe that the prolongation functor is a quasi-coherent sheaf of rings in the étale topology.

In section 2, we discuss the proof of Kolchin's theorem:

**Theorem 1** (Kolchin's irreducibility theorem - [14] Chapter IV, Proposition 10)**.** *Let $A$ be an integral domain, of finite type over a differential field $k$; then the associated differential variety is irreducible.*

The key point in the proof given here is that discrete valuation rings containing a field of characteristic zero are formally smooth over that field, and hence have rings of differential polynomials where are integral domains. It is interesting to notice that for a more general valuation ring it is still true that the associates ring of differential polynomials is an integral domain, but that the only proof that I know of this result uses Zariski's uniformization theorem.

In section 3, we turn to the result of Buium:

**Theorem 2** ([4], [3])**.** *Let $X$ be a variety, proper over an algebraically closed field $K$. Then $X$ is defined over the fixed field of the set of all derivations of $K$ which lift to derivations of the structure sheaf of $X$.*

The main tools in the proof given here are Grothendieck's theorem on algebraizability of morphisms between projective varieties over formal schemes, and Artin's approximation theorem.

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*Date*: May 10, 2001.

1991 *Mathematics Subject Classification.* Primary 12H05; Secondary 14.

*Key words and phrases.* Differential Algebra, irreducibility theorem.

Partially Supported by NSF grant DMS 98-01219.

Finally in section 4, two questions which arise from the techniques used in the paper are posed.

The initial genesis for this paper was a seminar at UIC organized by David Marker, Lawrence Ein and myself, in which we and some graduate students read the book [2] of Buium. In addition to my talk at the Rutgers Newark workshop, I gave talks on this material at the conference on model theory, algebraic and arithmetic at MSRI in 1989, and Columbia University and CCNY in 1999.

I would like to thank David Marker, Lawrence Ein, and Phyllis Cassidy, for discussions about various aspects of this work.

## 1. Differential Rings

### 1.1. **Some Commutative Algebra.**

1.1.1. *Étale homomorphisms.* See [19] for details on this section.

**Definition 1.** *Recall that a ring homomorphism $R \to S$ is* formally étale *if, given a ring $C$ and a square zero ideal $I \lhd C$, together with a commutative diagram of ring homomorphisms:*

$$\begin{tikzcd} S \arrow[r] & C/I \\ R \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

*there is a* unique *homomorphism $S \to C$ making the diagram commute. If instead we have that there exists* at least one *such homomorphism, we say that $R \to S$ is* formally smooth*, while if we have that there exists* at most one *such homomorphism, we say that $R \to S$ is* formally unramified*. If in addition to satisfying one of the above conditions, $S$ is an $R$-algebra of finite type, we remove the adjective "formally" and say that the morphism is* étale, smooth*, or* unramified *as appropriate.*

The following two exercises are straightforward.

**Exercise 1.** The composition of two formally étale, (resp. unramified, resp. smooth) maps is again formally étale, (resp. unramified, resp. smooth).

**Exercise 2.** If $S = R_t$ the localization of $R$ at a single element $t \in R$, then the localization map $R \to S_t$ is étale. (The key point is that if $I \lhd C$ is a nilpotent ideal, then an element $x \in C$ is a unit if and only if it is a unit modulo $I$.)

1.1.2. *Derivations.* If $R$ is a commutative ring, and $M$ an $R$-module, then recall that a derivation $\delta : R \to M$ is a map $\delta : R \to M$, such that $\delta(ab) = a\delta(b) + b\delta(a)$.

To give a derivation $\delta : R \to M$ is equivalent to giving a homomorphism of rings

$$\delta_* : R \to R \oplus M[\varepsilon]$$
$$\delta_* : r \longmapsto r + \delta(r)\varepsilon$$

where $R \oplus M[\varepsilon]$, with $\varepsilon^2 = 0$, is the ring of dual numbers over $R$ with coefficients in $M$, *i.e.* as an abelian group $R \oplus M[\varepsilon]$ is just the direct sum $R \oplus M$, and the multiplication law is defined by $(r + m\varepsilon).(r' + m'\varepsilon) = (rr' + (rm' + r'm)\varepsilon)$.

The set $\mathcal{D}er(R, M)$, of all derivations $\delta : R \to M$ is an $R$-module, indeed a submodule of the module of all functions from $R$ (viewed as a set) to $M$ (viewed as an $R$-module), with addition and scalar multiplication defined pointwise. The set

$\{r|\delta(r)=0\}$ is clearly a subring if $R$ (resp. subfield if $R$ is a field) which is called the *ring (resp. field) of constants* of $\delta$. If $R$ is a $k$-algebra, *i.e.,* there is a homomorphism $\phi : k \to R$, we write $\mathcal{D}er_k(R, M)$ for the submodule of $\mathcal{D}er(R, M)$ consisting of all $\delta$ for which $\delta(\phi(x)) = 0$ for all $x \in k$; note that $\mathcal{D}er(R, M) = \mathcal{D}er_{\mathbb{Z}}(R, M)$.

The covariant functor $M \mapsto \mathcal{D}er_k(R, M)$ from $R$-modules to $R$-modules is representable, by the $R$-module $\Omega_{R/k}$ of differentials of $R$ over $k$, [17]§25. *I.e.* $\mathcal{D}er_k(R, M) \simeq \mathrm{Hom}_{\mathrm{R}}(\Omega_{R/k}, M)$

If $R$ is a $k$-algebra, $f : R \to S$ is a homomorphism of $k$-algebras, and $M$ is an $S$-module, there is an exact sequence of $S$-modules:

$$0 \to \mathcal{D}er_R(S, M) \to \mathcal{D}er_k(S, M) \to \mathcal{D}er_k(R, M)$$

which is obtained by applying the functor $\mathrm{Hom}(\_, M)$ to the natural exact sequence:

$$S \otimes_R \Omega_{R/k} \overset{v_{S/R/k}}{\to} \Omega_{S/k} \to \Omega_{S/R} \to 0$$

**Proposition 1.** *If $S$ is formally smooth over $R$, the map $v_{S/R/k}$ has a left inverse, and the sequence becomes split exact, and hence the sequence 1.1.2 is also split exact for any $S$ module $M$.*

*Proof.* EGA 0, Théorème 20.5.7, ([10]). □

The reader may observe that there is a resemblance between the definitions, via lifting properties, of a formally smooth morphism and of a projective module. This is not a total coincidence:

**Proposition 2.** *Let $A$ be a ring, $V$ a projective $A$-module. Then the symmetric algebra $B = \mathbb{S}_A(V)$ is formally smooth over $A$,*

*Proof.* This follows immediately from the fact that $\mathrm{Hom}_{A-\mathrm{algebras}}(\mathbb{S}_A(V), C) \simeq \mathrm{Hom}_{A-\mathrm{modules}}(V, C)$. So that a lifting exists, even when the ideal $I \subset C$ is not nilpotent. □

**Proposition 3.** *Let $A$ be a ring, $B$ an $A$-algebra. Then if $B$ is formally smooth over $A$, $\Omega_{B/A}$ is a projective (but not necessarily finitely generated) $B$-module.*

*Proof.* EGA IV, Proposition 16.10.2, ([11]) - where this is deduced from EGA 0, Corollary 19.5.4, ([10]). □

### 1.1.3. *Definition and Elementary Properties.*

**Definition 2.** *A* differential ring *consists of a pair $(R, \delta)$ in which $R$ is a commutative ring with unit, and $\delta : R \to R$ is a derivation.*

If $(R, \delta)$ and $(R', \delta')$ are differential rings, then a ring homomorphism $f : R \to R'$ is said to be a *differential* homomorphism if it is compatible with the two derivations, *i.e.* if $\delta' \cdot f = f \cdot \delta$. The kernel of a differential homomorphism $(R, \delta) \to (R', \delta')$ is a *differential ideal, i.e.* an ideal $\mathfrak{p} \lhd R$ which is closed under $\delta$. Note that the intersection of two differential ideals in a differential ring $(R, \delta)$ is again a differential ideal, and hence any subset $X \subset R$ is contained in a smallest differential ideal, which we denote $[X]$.

Recall that to give a derivation $\delta : R \to R$ is equivalent to giving a homomorphism of rings

$$\begin{aligned}\delta_* &= \exp_{\leq 1}(\delta) : R \to R[\varepsilon] \\ \delta_* &= \exp_{\leq 1}(\delta) : r \longmapsto r + \delta(r)\varepsilon\end{aligned}$$

where $R[\varepsilon]$, with $\varepsilon^2 = 0$, is the ring of dual numbers over $R$. (The reason for the exponential notation for this map will become clearer later.) This interpretation leads to the following lemma:

**Lemma 1.** *Let $f : R \to S$ be a formally étale ring homomorphism. Then if $\delta_R : R \to R$ is a derivation, there is a* unique *derivation $\delta_S : S \to S$ extending $\delta$. Furthermore if $\sigma : S \to \Lambda$ is a ring homomorphism, with $\Lambda = (\Lambda, \delta_\Lambda)$ a differential ring, such that $\rho = \sigma \cdot f : R \to \Lambda$ is a differential homomorphism, then $\sigma$ is a differential homomorphism with respect to $\delta_S$ and $\delta_\Lambda$.*

*Proof.* Applying the definition of étale to the diagram

$$\begin{tikzcd} S \arrow[r, equal] & S \\ R \arrow[u] \arrow[r, "\delta_*"] & S[\epsilon] \arrow[u] \end{tikzcd}$$

where the bottom arrow is the composition $R \xrightarrow{\delta_*} R[\varepsilon] \subset S[\varepsilon]$ gives the first assertion. The second assertion follows by considering the diagram

$$\begin{tikzcd} S \arrow[r, "\sigma"] & \Lambda \\ R \arrow[u, "f"] \arrow[r, "\rho\cdot\delta_R"] & \Lambda[\epsilon] \arrow[u] \end{tikzcd}$$

There are two possible ways of filling this diagram in with a homomorphism from $S$ to $\Lambda[\varepsilon]$:

1. $\exp_{\leq 1}(\delta_\Lambda) \cdot \sigma$
2. $\rho \cdot \exp_{\leq 1}(\delta_R)$ (here we also use $\rho$ to denote the induced map of truncated polynomial rings).

Since $f$ is étale, and in particular unramified, these maps must be equal, and we are done. ☐

Since localization is étale, we automatically get:

**Corollary 1.** *If $R = (R, \delta_R)$ is a differential ring, and $S \subset R$ is a multiplicative set, the derivation $\delta_R$ has a unique extension to the localization $S^{-1}R$.*

**Lemma 2.** *If $R$ is a differential ring, then its nilradical is a differential ideal.*

*Proof.* This is a standard result, which may be found in [2] for example. The key point is that it is easy to see, using induction on $n$, that for any $a \in R$ and any $n \in \mathbb{N}$, $\delta(a^n) - \delta(a)^n$ is divisible by $a$. Hence if $a^n = 0$, we have that $\delta(a)^n$ is divisible by $a$ and is therefore nilpotent - and hence $\delta(a)$ itself is nilpotent. ☐

**Lemma 3.** *If $R$ is a differential ring, and $\mathfrak{p} \lhd R$ is a minimal prime ideal, then $\mathfrak{p}$ is a differential ideal.*

*Proof.* $\mathfrak{p}$ is the inverse image of $\mathfrak{p}R_{\mathfrak{p}}$ under the natural map $R \to R_{\mathfrak{p}}$. The derivation in $R$ extends to $R_{\mathfrak{p}}$, and $\mathfrak{p}R_{\mathfrak{p}}$ is the unique minimal prime ideal in $R \to R_{\mathfrak{p}}$ - hence is equal to the nilradical of $R_{\mathfrak{p}}$, and is therefore a differential ideal in $R_{\mathfrak{p}}$. Since the inverse image of a differential ideal is a differential ideal, it follows that $\mathfrak{p}$ is a differential ideal. □

## 1.2. **Prolongation.**

1.2.1. *Existence.* If $(k, \delta)$ is a differential ring we may consider the category $\mathbf{Diff}_{(k,\delta)}$ of differential $(k, \delta)$-algebras. There is clearly a forgetful functor $U : \mathbf{Diff}_{(k,\delta)} \to \mathbf{Alg}_k$, which associates to the differential $(k, \delta)$-algebra $(R, \tilde{\delta})$ the $k$-algebra $R$.

**Lemma 4.** *The forgetful functor* $U : \mathbf{Diff}_{(k,\delta)} \to \mathbf{Alg}_k$ *has a left adjoint.*

*Proof.* It is a general fact in Universal algebra that the forgetful functor between two categories of algebras induced by forgetting one or more operations or equations has a left adjoint. More generally, it is a result of Lawvere that "algebraic functors" have left adjoints. It is difficult to give a nice reference for the proof, in part because of variations in the way that a *variety of algebras* can be defined. However one can find, in various references, proofs of the existence of free algebras, i.e. of left adjoints for the forgetful functors from a category of algebras, such as differential $k$-algebras or $k$-algebras, to the category of sets. For example, see [15] section V.6, and in particular the discussion following Theorem 3. The more general case of the lemma then follows immediately from the existence of free algebras via theorem 28.12 in [13]. □

Let us denote the left adjoint of the Lemma

$$(\ )^\infty : R \mapsto R^\infty$$

Thus if $(k, \delta = \delta_k)$ is a differential ring, $(\ )^\infty$ associates to each $k$-algebra $R$ a differential $(k, \delta)$-algebra $R^\infty$, and a ring homomorphism $\eta :\to R^\infty$with the following universal property: given a homomorphism $\phi : (k, \delta) \to (S, \delta)$ of differential rings, and a ring homomorphism $f : R \to S$, there is a unique homomorphism $f^\infty : R^\infty \to S$ of differential rings which makes the following diagram commute:

$$\begin{tikzcd}
R^\infty \arrow[r, "f^\infty"] & S \\
R \arrow[u, "\eta"] \arrow[r, "f"] & S \arrow[u, equal] \\
k \arrow[u] \arrow[r, equal] & k \arrow[u]
\end{tikzcd}$$

If $\delta_k = 0$, then $R^\infty$ is the coordinate ring of the infinite jet bundle over $\mathrm{Spec}(R)$. Thus the prolongation $\mathrm{Spec}(R^\infty)$ of $\mathrm{Spec}(R)$ is a "twisted" jet bundle.

**Lemma 5.** *The functor* $(\ )^\infty : \mathbf{Alg}_k \to \mathbf{Diff}_{(k,\delta)}$ *commutes with direct (or inductive) limits.*

*Proof.* This is a standard property of left adjoints. See [15] V.6 for example. □

1.2.2. *Explicit Constructions.* Let's look at two explicit constructions of $R^\infty$. The first of these constructions also provides an alternative proof of the existence of $R^\infty$. Start by observing that if $(k,\delta)$ is a differential ring, and $X$ is a set, we can form (by the existence of free algebras) the *free* $(k,\delta)$*-algebra* $(k,\delta)\{X\}$, usually called the ring of differential polynomials over $k$, and simply written $k\{X\}$. This is the polynomial ring over $k$ on the set

$$X \cup (X \times \mathbb{N}) \simeq \coprod_{j \in 0 \cup \mathbb{N}} X = \{x^{(j)} | x \in X,\ j \in 0 \cup \mathbb{N}\}\ ,$$

with the derivation, extending that on $k$, determined by the rule that $x^{(0)} = x$ for all $x \in X$, and $\delta(x^{(j)}) = x^{(j+1)}$ for $j \geq 0$. It is easily checked that $k\{X\} = k[X]^\infty$, for if $(k,\delta) \to (S,\delta)$ is a homomorphism of differential rings, and $f : k[X] \to S$ is a ring homomorphism, it has a unique extension to $f^\infty : k\{X\} \to S$ by the rule $f^\infty(x^{(j)}) = \delta^j(f(x))$, where $\delta^j = \delta \circ \ldots \circ \delta$ is the $j$-fold composition of $\delta$ with itself.

Now *any* $k$-algebra $R$ can be realized as a quotient:

$$k[X] \twoheadrightarrow R$$

(for example, we can take $X = R$ itself), and so we have the following description of $R^\infty$:

**Lemma 6.** *Let $R$ be a $k$-algebra, with $(k,\delta)$ a differential ring, and suppose that $k[X] \twoheadrightarrow R$ is a surjective homomorphism, with domain a polynomial ring over $k$. If we write $\mathfrak{p}$ for the kernel of this homomorphism, so that $k[X]/\mathfrak{p} \simeq R$ then there is a canonical isomorphism:*

$$k\{X\}/[\mathfrak{p}] \twoheadrightarrow R^\infty$$

*Proof.* Exercise. □

We can also give another construction of $R^\infty$ using the universal properties of the module of differentials, which shows that when $R$ is a formally smooth $k$-algebra which is an integral domain, then the same is true of $R^\infty$.

**Proposition 4.** *Suppose that $R$ is formally smooth over the differential ring $k$. Then there is a sequence of rings $R^{\{n\}}$, for $n \geq -1$, with the following properties:*

1. *$R^{\{-1\}} = k$.*
2. *$R^{\{0\}} = R$.*
3. *For all $n \geq -1$, there is a ring homomorphism $R^{\{n\}} \to R^{\{n+1\}}$.*
4. *For each $n \geq -1$, there is a derivation $\delta^{\{n\}} : R^{\{n\}} \to R^{\{n+1\}}$ such that we have a commutative diagram:*

$$\begin{tikzcd} R^{\{n\}} \arrow[r, "\delta^{\{n\}}"] & R^{\{n+1\}} \\ R^{\{n-1\}} \arrow[u] \arrow[r, "\delta^{\{n-1\}}"] & R^{\{n\}} \arrow[u] \end{tikzcd}$$

5. *There is a sequence which is universal (*i.e.* an initial object in the appropriate category) among all sequences with properties (1)-(4). For this universal sequence, $R^{\{n+1\}}$ is formally smooth over $R^{\{n\}}$, indeed it is isomorphic to the symmetric algebra on a projective $R^{\{n\}}$ module. Therefore, if $R$ is an integral domain, so is $R^{\{n\}}$ for all $n \geq 0$, and hence $R^\infty$ is an integral domain.*

*Proof.* We proceed by induction on $n \geq 0$. The case $n = 0$ is already given to us. Therefore, given $R^{\{n\}}$, together with a derivation $R^{\{n-1\}} \to R^{\{n\}}$, so that the sequence $k \to R \to R^{\{1\}} \to \dots R^{\{n\}}$ is universal, in order to construct a $R^{\{n\}}$ algebra which satisfies 4) and is universal among all such, it suffices to prove the following lemma: □

**Lemma 7.** *Suppose that $A$ is a commutative ring, that $B$ is an $A$-algebra, and that we are given a derivation $\delta : A \to B$. The functor from the category of $B$-algebras to the category of sets, which assigns to the $B$-algebra $R$ the set of all derivations $\tilde{\delta} : B \to R$ which extend the derivation $\delta$ is representable, and if $B$ is formally smooth over $A$, the representing object is isomorphic to the symmetric algebra on a projective $B$-module.*

*Proof.* If we omit the requirement that $\tilde{\delta} : B \to R$ extend $\delta$, then we are simply representing the set of all derivations from $B$ to $R$. The set of such derivations is represented by the set of $B$-module homomorphisms $\Omega_B \to R$. Since $R$ is a $B$-algebra, this is equivalent to the set of all $B$-algebra homomorphisms $\mathbb{S}_B(\Omega_B) \to R$. The condition that $\tilde{\delta} : B \to R$ extend the derivation $\delta$ is that the homomorphism $\mathbb{S}_B(\Omega_B) \to R$ map all elements of the form $da$ for $a \in A$ to $\delta(a)$. Thus we divide $\mathbb{S}_B(\Omega_B)$ by the ideal $I$ generated by all elements of the form $da - \delta(a)$ for $a \in A$. If we now assume that $B$ is formally smooth over $A$, then we know that $\Omega_B$ is non-canonically isomorphic to the direct sum of $\Omega_{B/A}$ and $\Omega_A$. Therefore the quotient $\mathbb{S}_B(\Omega_B)/I$ is isomorphic to $\mathbb{S}_B(\Omega_{B/A})$. □

**Remark 1.** *If we do not assume that $R$ is formally smooth over $k$, then the Proposition remains true, except that in part (5), only the first sentence would hold.* I.e., *there exists a universal sequence of algebras and morphisms, but the $R^{\{n\}}$ are not in general formally smooth or integral domains.*

We immediately get from property (5) in the proposition:

**Corollary 2.** *If $(k, \delta)$ is a differential ring, and $R$ is a formally smooth $k$-algebra, which is an integral domain, then $R^\infty$ is an integral domain, and $R \to R^\infty$ is injective.*

In particular, by a theorem of Cohen, we know that if $k \subset K$ is an extension of fields of characteristic zero, then $K$ is formally smooth over $k$, (see [10]) Ch. 0, §Théorème 19.6.1) and hence if $k$ a differential field, then in the category of $k$-algebras, $K^\infty$ is an integral domain.

### 1.2.3. *Etale Base Change.*

**Lemma 8.** *Let $(k, \delta)$ be a differential ring. Suppose that $f : R \to S$ is a formally étale homomorphism of $k$-algebras. Then*

$$S^\infty \simeq S \otimes_R R^\infty$$

*Proof.* Suppose that we are given a $(k, \delta)$ algebra $(\Lambda, \delta)$ and ring homomorphism $\phi : S \to \Lambda$. We wish to show that this has a canonical factorization through a differential homomorphism $S \otimes_R R^\infty \to \Lambda$. Writing $i : R \to S$ for the inclusion, the induced homomorphism $\phi \cdot i : R \to \Lambda$ has a canonical factorization $\phi \cdot i = (\phi \cdot i)^\infty \cdot \eta_R$, where $(\phi \cdot i)^\infty : R^\infty \to \Lambda$ is a differential homomorphism. Hence we have a commutative diagram:

$$\begin{tikzcd}
S \arrow[r, "\phi"] & \Lambda \\
R \arrow[u, "i"] \arrow[r, "\eta_R"] & R^\infty \arrow[u, "(\phi\cdot i)^\infty"']
\end{tikzcd}$$

and a canonical map of commutative rings

$$S \otimes_R R^\infty \to \Lambda$$

since $R^\infty \to S \otimes_R R^\infty$ is étale and $R^\infty \to \Lambda$ is a differential homomorphism. It then follows from Lemma 1 that the induced map $S \otimes_R R^\infty \to \Lambda$ is a differential homomorphism. The rest of the details are left to the reader. ☐

**Definition 3.** *The conclusion of the lemma could be phrased as "the ring of differential polynomials is a quasi-coherent sheaf in the étale topology". Thus we can define,* for any scheme $X$, *the sheaf (in the étale topology)* $\mathcal{O}_X^\infty$ *of differential polynomials on* $X$, *and if* $X = \operatorname{Spec}(R)$ *is affine, then*

$$\Gamma(X, \mathcal{O}_X^\infty) \simeq R^\infty$$

**Definition 4.** *Let* $k = (k, \delta)$ *be a differential ring. Suppose that* $X$ *is scheme over* $k$. *Then we define* $X^\infty := \textbf{\textit{Spec}}(\mathcal{O}_X^\infty)$. *(See* [12] *Ch. II, exercise 5.17 for the construction of* ***Spec****.)* $X^\infty$ *is a "differential scheme" in the sense that* $\mathcal{O}_{X^\infty}$ *is equipped with a derivation. The passage from* $X$ *to* $X^\infty$ *is known as* prolongation*; when the derivation* $\delta$ *on* $k$ *is zero,* $X^\infty$ *is the (infinite) jet bundle. However,* $X^\infty$ *is not a differential scheme in the sense of other authors* - i.e.*, it does* not *use the "differential spectrum" as a local model.*

**Corollary 3.** *If* $k = (k, \delta)$ *is a differential field, and* $k \subset K$ *is a finitely generated separable extension of* $k$*, then* $K^\infty$ *is a polynomial ring over* $K$.

*Proof.* By [6] Chapter V, §16.7 Corollary to Theorem 5, we know that $K$ has a separating transcendence basis $\{x_1, \ldots, x_n\}$ over $k$. Thus $K$ is a finite separable algebraic (and hence étale) extension of $k(x_1, \ldots, x_n)$. Since localizations (including passage to the field of fractions) are formally étale, it follows that $K$ is formally étale over the polynomial ring $k[x_1, \ldots, x_n]$. Hence, by the lemma, $K^\infty \simeq k\{x_1, \ldots, x_n\} \otimes_{k[x_1,\ldots,x_n]} K \simeq K[\{x_1, \ldots, x_n\} \times \mathbb{N}]$. ☐

**Corollary 4.** *If we assume in addition that* $k = (k, \delta)$ *above is characteristic zero, then we can remove the assumption of finite generation.*

*Proof.* By Steinitz theorem ([6] Chapter V, theorem 1 of §14.2) $K$ has a transcendence basis $X \subset K$ over $k$, and the extension $k(X) \subset K$ is algebraic, hence (since we assume characteristic zero) a composite of finite separable extensions, and hence formally étale. Then as in the previous corollary, we get that $K^\infty$ is a polynomial ring over $K$. (See also *op. cit.* V.16.3.) ☐

Lemma 8 gives another proof of Corollary 2 in the case when $R$ is a smooth (not just formally smooth) $k$-algebra.

**Corollary 5.** *If $R$ is a smooth $k$-algebra, with $R$ an integral domain, then $R^\infty$ is an integral domain.*

*Proof.* Since $R$ is smooth over $k$, $X = \mathrm{Spec}(R)$ is locally isomorphic in the étale topology to affine space $\mathbb{A}^n_k$ (where $n$ is the dimension of $R$ over $k$). Hence $R^\infty$ is locally isomorphic to the ring of differential polynomials $k\{x_1 \dots x_n\}$. Since flatness is a local condition, and any polynomial ring is flat over its ring of coefficients, $R^\infty$ is flat over $R$. Since $R$ is a domain, it injects into its fraction field $F$. Since $R^\infty$ is flat over $R$, we have that $R^\infty$ injects into $R^\infty \otimes_R F$. But we know that $R \mapsto R^\infty$ commutes with localization, hence $R^\infty \otimes_R F \simeq F^\infty$, which by 3 is a polynomial ring over $F$, and is therefore a domain. Thus $R^\infty$ injects into a domain, and is itself a domain. ☐

**Corollary 6.** *If $R = (R, \delta_R)$ is a differential ring, then every minimal prime ideal in $R$ is a differential ideal.*

*Proof.* Suppose that $\mathfrak{p} \lhd R$ is a minimal prime. By the preceding Corollary, the differential $\delta_R$ has a unique extension to $R_\mathfrak{p}$, with respect to which the localization map is a differential homomorphism. Since $\mathfrak{p}$ is minimal, $\mathfrak{p}R_\mathfrak{p}$ is a minimal prime ideal in $R_\mathfrak{p}$ (Thus $R_\mathfrak{p}$ is Artinian.) Since the prime ideals of $R_\mathfrak{p}$ are in 1-1 (inclusion preserving) correspondence with the primes of $R$ containing $\mathfrak{p}$, it follows that $\mathfrak{p}R_\mathfrak{p}$ is the unique minimal prime ideal in $R_\mathfrak{p}$, and is therefore the nilradical of $R_\mathfrak{p}$. Hence, by Lemma 2, we know that it is a differential ideal. Since the inverse image of a differential ideal by a differential homomorphism is certainly a differential ideal, we are done. ☐

## 2. Kolchin's Irreducibility Theorem

In the last section we saw that if $X$ is a variety smooth and integral over a differential field of characteristic zero, then the differential algebraic variety $X^\infty$ associated by prolongation to $X$ is integral, and in particular irreducible. Kolchin's Irreducibility Theorem tells us that in characteristic zero, the irreducibility conclusion remains true even if $X$ is not smooth.

### 2.1. **Preliminary Lemmas.** We start by considering the key case of valuation rings.

**Lemma 9.** *Let $k$ be a differential field of characteristic zero, and $E \subset F$ an extension of fields containing $k$-algebras. Then $E^\infty \to F^\infty$ is injective.*

*Proof.* Since prolongation commutes with direct limits, and direct limits preserve injections, we may assume that $E$ and $F$ are finitely generated extensions of $k$, in which case the lemma follows immediately from Corollaries 3 and 4.

☐

**Lemma 10.** *Let $k$ be a field of characteristic zero, and $R$ a $k$-algebra which is a discrete valuation ring. Then $R$ is a formally smooth $k$ algebra.*

*Proof.* Let $K$ be the residue field of $R$. The induced map from $k$ to $K$ is an inclusion, which makes $K$ a separable extension of $k$, and hence using Cohen's theorem again, (see EGA Ch.. 0, Corollary §19.6.1, in [10]) $K$ is formally smooth over $k$. If $M$ is the maximal ideal of $R$, then $M/M^2 \simeq K$ and is therefore a projective $K$-module. Therefore by Corollary §19.5.4, in *op. cit.*, $R$ is formally smooth over $k$. ☐

From the Lemma, together with Proposition 4 we immediately get:

**Corollary 7.** *Let $k$ be a differential field of characteristic zero, and $R$ a $k$-algebra which is a discrete valuation ring. Then $R^\infty$ is an integral domain.*

More generally, by using uniformization, we have:

**Lemma 11.** *Let $k$ be a differential field of characteristic zero, and $R$ a $k$ algebra which is a valuation ring. Then $R^\infty$ is an integral domain.*

*Proof.* By the *uniformization theorem* of Zariski, [20], see also Popescu [18] for a more general result, we know that any valuation ring $R$ containing a field $k$ of characteristic zero is the direct limit of smooth $k$-algebras (which we may assume to be integral domains):

$$R = \varinjlim_\alpha A_\alpha \ .$$

Since $R \mapsto R^\infty$ commutes with direct limits, we have that

$$R^\infty = \varinjlim_\alpha (A^\alpha)^\infty$$

is a direct limit of integral domains, and is therefore an integral domain. □

## 2.2. Statement of the Theorem.

**Theorem 3.** *Let $k = (k, \delta)$ be a differential field of characteristic zero, and suppose that $R$ is a $k$-algebra which is an integral domain. Then $R^\infty$ has a unique minimal prime ideal.*

The following definition will be useful in the proof of the theorem:

**Definition 5.** *Let $R$ be a $k$-algebra, with $k$ a differential field of characteristic zero. Suppose that $\mathfrak{p}$ is a prime ideal in $R$. Denote by $[[\mathfrak{p}]]$ the prime differential ideal in $R^\infty$ which is the kernel of the homomorphism $R^\infty \to k(\mathfrak{p})^\infty$ where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$. Note that $[[\mathfrak{p}]]$ is a prime ideal because $k(\mathfrak{p})^\infty$ is an integral domain by Corollary 4.*

With this definition, we can rephrase Kolchin's theorem as asserting that any prime ideal $\mathfrak{p} \lhd R^\infty$ contains $[[0]]$.

## 2.3. Proof of the theorem.

We start by assuming that $R$ is finitely generated over $k$. The proof in the noetherian case then consists of three steps:

1. Any prime ideal $\mathfrak{p} \lhd R^\infty$ contains a prime differential ideal. Therefore it is enough to show that any prime *differential* ideal $\mathfrak{p} \lhd R^\infty$ contains $[[0]]$.
2. If $\eta : R \to R^\infty$ is the natural map, then for any prime differential ideal $\mathfrak{p} \lhd R^\infty$, $[[\eta^{-1}(\mathfrak{p})]] \subset \mathfrak{p}$. Therefore it is enough to prove that for any prime ideal $\mathfrak{q} \lhd R$, $[[0]] \subset [[\mathfrak{q}]] \lhd R^\infty$.
3. If $R$ is a one dimensional local domain, with maximal ideal $\mathfrak{p}$, then $[[0]] \subset [[\mathfrak{p}]] \lhd R^\infty$.
4. If $\mathfrak{p} \subset \mathfrak{q}$ are prime ideals in $R$, with $\mathfrak{p}$ of height one above $\mathfrak{p}$, then $[[\mathfrak{p}]] \subset [[\mathfrak{q}]] \lhd R^\infty$
5. If $\mathfrak{p} \subset \mathfrak{q}$ are arbitrary prime ideals in $R$, then $[[\mathfrak{p}]] \subset [[\mathfrak{q}]] \lhd R^\infty$. In particular for any prime ideal $\mathfrak{q} \lhd R$, $[[0]] \subset [[\mathfrak{q}]] \lhd R^\infty$.

2.3.1. *Proof of Step 1.* By Zorn's lemma any prime ideal contains a minimal prime ideal. However, since we are assuming that $R$ is finitely generated over $k$, and therefore noetherian, we can also use the fact that by Krull's principal ideal theorem, (Theorem §13.5 of [17] or section 12.E of [16]) the primes ideals in $R$ satisfy the descending chain condition.

2.3.2. *Proof of Step 2.*

**Lemma 12.** *Let $k$ and $R$ be as above. Suppose that $\mathfrak{p}$ is a prime differential ideal in $R^\infty$. If we write $\mathfrak{p}_0$ for the prime ideal $\eta^{-1}\mathfrak{p}$, where $\eta$ is the natural map from $R$ to $R^\infty$, then $[[\mathfrak{p}_0]] \subset \mathfrak{p}$.*

*Proof.* Since the map $R \to R^\infty/\mathfrak{p}$ factors through $R/\mathfrak{p}_0$, there is an induced map $\theta : (R/\mathfrak{p}_0)^\infty \to R^\infty/\mathfrak{p}$, and the quotient map $R^\infty \to R^\infty/\mathfrak{p}$ factors through $\theta$. Since the prolongation functor $(\ )^\infty$ commutes with localization, if $k$ is the residue field of $R/\mathfrak{p}_0$, the natural map $R^\infty \otimes_R k \to k^\infty$ is an isomorphism. Therefore there is a commutative diagram:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R^\infty \arrow[r, equal] \arrow[d] & R^\infty \arrow[d] \\
(R/\mathfrak{p}_0) \arrow[r] \arrow[d] & (R/\mathfrak{p}_0)^\infty \arrow[r] \arrow[d] & R^\infty/\mathfrak{p} \arrow[d] \\
k \arrow[r] & k^\infty \simeq (R/\mathfrak{p}_0)^\infty \otimes_R k \arrow[r] & R^\infty/\mathfrak{p} \otimes_{R/\mathfrak{p}_0} k
\end{tikzcd}$$

Since the bottom left hand vertical map in this diagram is a localization, the other two bottom vertical maps are also localizations. Therefore, since $R^\infty/\mathfrak{p}$ is an integral domain, the right bottom vertical map is injective. Hence:

$$\operatorname{Ker}(R^\infty \to R^\infty/\mathfrak{p}) = \operatorname{Ker}(R^\infty \to R^\infty/\mathfrak{p} \otimes_{R/\mathfrak{p}_0} k)$$

and therefore:

$$[[\mathfrak{p}_0]] = \operatorname{Ker}(R^\infty \to k^\infty) \subset \mathfrak{p} = \operatorname{Ker}(R^\infty \to R^\infty/\mathfrak{p} \otimes_{R/\mathfrak{p}_0} k)$$

□

2.3.3. *Proof of step 3.*

**Lemma 13.** *Let $k$ and $R$ be as above, and assume in addition that $R$ is a one dimensional noetherian local domain. If $\mathfrak{p}$ is the maximal ideal in $R$, then with the notation above,*

$$[[0]] \subset [[\mathfrak{p}]]$$

*Proof.* Let $A$ be the normalization (=integral closure in its fraction field) of $R$. By [5], V§2.1, Proposition 1, for any maximal ideal $\mathfrak{m} \lhd A$, $\mathfrak{m} \cap R = \mathfrak{p}$. Then by the Krull-Akizuki theorem, (Proposition 5 of Chapter VII, §2.5, ([5])), $A$ is a Dedekind domain. Hence for any maximal ideal $\mathfrak{m} \lhd A$, $A_\mathfrak{m}$ is a discrete valuation ring.

Therefore, we have a commutative diagram:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R/\mathfrak{p} = k_0 \arrow[d] \\
A \arrow[r] \arrow[d] & A/\mathfrak{m} = k \\
F &
\end{tikzcd}$$

in which all the vertical maps are injective, with $k_0 \subset k$ a separable field extension, and $A \subset F$ a localization. Since $A$ is a discrete valuation ring, $A^\infty$ is an integral domain, by Lemma 10, and $A^\infty \to F^\infty = A^\infty \otimes_A F$ is injective. Therefore $[[0]] = \mathrm{Ker}(R^\infty \to F^\infty) = \mathrm{Ker}(R^\infty \to A^\infty)$. Since $k_0 \to k$ is injective and $k$ is characteristic zero, the field extension $k|k_0$ is formally smooth, and finite degree by the Krull-Akizuki theorem, hence étale. Therefore, by étale base change, $k^\infty \simeq (k_0)^\infty \otimes_{k_0} k$ and so the map $(k_0)^\infty \to k^\infty$ is injective. Therefore $[[\mathfrak{p}]] = \mathrm{Ker}(R^\infty \to k^\infty)$. Since the map $R^\infty \to k^\infty$ factors through $A^\infty$, we get that $[[0]] \subset [[\mathfrak{p}]]$. □

### 2.3.4. *Proof of step 4.*

**Lemma 14.** *Suppose that* $\mathfrak{p} \subset \mathfrak{b} \lhd R$ *when* $\mathfrak{b}$ *has height 1 above* $\mathfrak{p}$. *Then* $[[\mathfrak{p}]] \subset [[\mathfrak{b}]] \lhd R^\infty$

*Proof.* We know by the previous lemma that, that since $R_\mathfrak{b}/(\mathfrak{p}R_\mathfrak{b}) \simeq (R/\mathfrak{p})_\mathfrak{b}$ is a one dimensional local ring, there is an inclusions of kernels:

$$\mathrm{Ker}((R_\mathfrak{b}/(\mathfrak{p}R_\mathfrak{b}))^\infty \to k(\mathfrak{p})^\infty) \subset \mathrm{Ker}((R_\mathfrak{b}/(\mathfrak{p}R_\mathfrak{b}))^\infty \to k(\mathfrak{b})^\infty) .$$

Hence there is also an inclusion,

$$\mathrm{Ker}(R^\infty \to k(\mathfrak{p})^\infty) \subset \mathrm{Ker}(R^\infty \to k(\mathfrak{b})^\infty)$$

as desired. □

2.3.5. *Proof of step 5.* As has already been remarked, since we are assuming that $R$ is noetherian, primes ideals in $R$ satisfy the descending chain condition. In particular, if $\mathfrak{p} \lhd R$ is a prime ideal, there are only finitely many prime ideals between $0 \lhd R$ and $\mathfrak{p}$, and so step 5 follows by induction from step 4.

2.3.6. *Remark.* If we had not assumed that $R$ is noetherian, then we could have replaced steps 3 and 5 with the following argument:

**Lemma 15.** *Let* $\Lambda$ *be a differential ring, and* $R$ *be a* $\Lambda$*-algebra which is a local domain. If* $\mathfrak{p}$ *is the maximal ideal in* $R$, *then with the notation above,*

$$[[0]] \subset [[\mathfrak{p}]]$$

*Proof.* Now, rather than using integral closure and the Krull-Akizuki theorem, we shall use Lemma 11 (which uses uniformization), together with the existence of valuations with a given center.

Let $F$ be the fraction field of $R$. By [5], Chapter VI, 1.2 Corollary, there is a valuation ring $A \subset F$ which dominates $R$, *i.e.*, $R \subset A$, and if $\mathfrak{m} \lhd A$ is the maximal ideal of $A$, $\mathfrak{m} \cap R = \mathfrak{p}$. Then, we have a commutative diagram:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R/\mathfrak{p} = k_0 \arrow[d] \\
A \arrow[r] \arrow[d] & A/\mathfrak{m} = k \\
F &
\end{tikzcd}$$

in which all the vertical maps are injective, and $A \subset F$ a localization. Since $A$ is a valuation ring, $A^\infty$ is an integral domain, by Lemma 11, and $A^\infty \to F^\infty = A^\infty \otimes_A F$ is injective. Therefore $[[0]] = \mathrm{Ker}(R^\infty \to F^\infty) = \mathrm{Ker}(R^\infty \to A^\infty)$. Since $k_0 \to k$ is injective and $k$ is characteristic zero, the map $(k_0)^\infty \to k^\infty$ is injective by Lemma 8. Therefore $[[\mathfrak{p}]] = \mathrm{Ker}(R^\infty \to k^\infty)$. Since the map $R^\infty \to k^\infty$ factors through $A^\infty$, we get that $[[0]] \subset [[\mathfrak{p}]]$. □

As we will see below, we do not need the case of general valuation rings to deduce Kolchin's theorem in the non-noetherian case.

2.3.7. *The non-noetherian case.* Suppose now that $R$ is an arbitrary integral domain in the category of $k$-algebras, *i.e.,* $R$ not necessarily finitely generated over $k$. Since the functor $(\ )^\infty$ commutes with direct limits, $R^\infty$ is the direct limit of $A^\infty$, as $A$ runs through the partially ordered set of finitely generated subalgebras $A \subset R$. Since each such subalgebra $A$ is an integral domain which is finitely generated over $k$, we know that each $A^\infty$ has a single minimal prime. Note that the assertion that a ring have a single minimal prime is equivalent to saying that its nilradical is prime. Suppose now that a pair of elements $a, b \in R^\infty$ have nilpotent product: $(ab)^n = 0$ for some $n \in \mathbb{N}$. Then there exists a finitely generated $A \subset R$ such that $a, b \in A^\infty$, and $(ab)^n = 0$ in $A^\infty$. By the finite type case of the theorem, it follows that one or other of $a$ or $b$ is nilpotent in $A^\infty$, and hence in $R^\infty$, and we are done.

Therefore the proof of Kolchin's theorem is complete.

## 3. Descent for Projective Varieties

3.1.

**Theorem 4.** *Let $X$ be a proper variety over an algebraically closed field $K$ of characteristic zero. Let $\Delta$ be the $K$-vector space $H^0(X, \mathcal{D}er(\mathcal{O}_X))$ of global sections of the sheaf of derivations of the structure sheaf of $X$. Let $K^\Delta$ be the (algebraically closed) subfield of $K$ consisting of elements fixed under the action of $\Delta$. Then there exists a variety $Y$, proper over $K^\Delta$, and an isomorphism*

$$X \simeq Y \otimes_{K^\Delta} K$$

i.e., *$K^\Delta$ is a field of definition for $X$. Furthermore, $K^\Delta$ is the minimal field of definition for $X$, in the sense that any other algebraically closed subfield $L \subset K$ which is a field of definition for $X$ contains $K^\Delta$.*

3.2. The proof of the theorem will be in several steps. Without loss of generality, we assume that $X$ is connected.

3.2.1. *Step 1.*

**Lemma 16.** *$K^\Delta$ is contained in all other fields of definition for $X$.*

(Hence it is enough to show that $X$ is defined over $K^\Delta$.)

*Proof.* Suppose that $F \subset K$ is a field of definition for $X$. Thus there is a variety $Y$, proper over $F$, and an isomorphism

$$X \simeq Y \otimes_F K$$

On the category of $F$-algebras, $A \to \Omega_{A/F}$ commutes with direct limits (see, for example, [7] theorems 16.5 and 16.8).

Thus we have an isomorphism of sheaves on $X \otimes_F K$:

$$\Omega_{(\mathcal{O}_Y \otimes_F K)/F} \simeq (\mathcal{O}_Y \otimes_F \Omega_{K/F}) \oplus (K \otimes_F \Omega_{\mathcal{O}_Y/F})$$

and therefore there is an isomorphism

$$H^0(X, \mathcal{D}er_F(\mathcal{O}_X)) \simeq \mathcal{D}er_F(K) \oplus (K \otimes_F H^0(Y, \mathcal{D}er_F(\mathcal{O}_Y))$$

It follows that $\mathcal{D}er_F(K) \subset \Delta = H^0(X, \mathcal{D}er(\mathcal{O}_X))$, and it therefore acts naturally on $K \simeq H^0(X, \mathcal{O}_X)$ with fixed field $F$, and hence $K^\Delta \subset F = K^{\mathcal{D}er_F(K)}$. □

3.2.2. *Step 2.*

**Lemma 17.** *There exists a field of definition $F$ for $X$ which is of finite transcendence degree over the prime field $\mathbb{Q}$.*

*Proof.* This is a standard argument: as a variety of finite type over $K$, $X$ may be defined by a finite set of equations. We may then take $F$ to be the algebraic closure of the subfield of $K$ generated by these equations. □

**Lemma 18.** *In order to show that $X$ may be defined over $K^\Delta$ it suffices to show that if there is a nonzero element $\xi \in H^0(X, \mathcal{D}er(\mathcal{O}_X))$, with $X$ defined over a field $K$ which is finitely generated over the prime field, then $X$ is defined over a field $F$ which has strictly smaller transcendence degree over the prime field than $K$ does.*

*Proof.* Since, by the previous lemma, there exist fields of definition for $X$ which have finite transcendence degree over $\overline{\mathbb{Q}}$, we know that we can choose a field of definition $F$ for $X$ which has minimal transcendence degree over $K^\Delta$. Since $F$ is a field of definition for $X$, we may write $X \simeq Y \otimes_F K$, with $Y$ projective over $F$, If $F \neq K^\Delta$, then by definition of $K^\Delta$ there exists a nonzero element $\delta \in H^0(Y, \mathcal{D}er(\mathcal{O}_Y))$ which has a non-trivial action on $F$. Hence if we can show that this implies that $Y$ is defined over a smaller field than $F$, this will contradict the minimality of $F$. □

3.2.3. *Step 3.*

**Proposition 5.** *Suppose that $X$ is proper and connected over an algebraically closed field $K$ of finite transcendence degree over $\overline{\mathbb{Q}}$, and that there is a $\delta \in H^0(X, \mathcal{D}er(\mathcal{O}_X))$, such that* $\mathrm{trdeg}(K/K^\delta) > 0$. *(Note that $\delta$ acts on $K = H^0(X, \mathcal{O}_X)$). Then there exists a subfield $F \subset K$, with* $\mathrm{trdeg}(K/F) > 0$ *such that $X$ is defined over $F$.*

In order to prove the proposition, we first need a lemma:

**Lemma 19.** *Let $X$ and $K$ be as above. Then there exists a discrete valuation ring $R \subset K$, such that:*

(1) *$R$ is a localization of an algebra which is smooth and of finite type over $\overline{\mathbb{Q}}$*
(2) *there is a scheme $\mathcal{X}$ which is proper, with geometrically connected fibres, over $S = \mathrm{Spec}(R)$ such that $\mathcal{X} \otimes_R K = X$.*
(3) *there is a $\xi \in H^0(\mathcal{X}, \mathcal{D}er(\mathcal{O}_{\mathcal{X}}))$ which restricts to $\delta \in H^0(X, \mathcal{D}er(\mathcal{O}_X))$.*
(4) *If we write $\overline{\xi}$ for the derivation of $R$ induced by $\xi$, and $f$ for the generator of the maximal ideal of $R$, then $\overline{\xi}(f)$ is a unit in $R$.*
(5) *$R$ contains a subfield $E$, such that the residue field $k$of $R$ is a finite algebraic extension of $E$ under the natural inclusion of $E \subset k$*

*Proof.* Arguing as in the proof of Lemma 17, we first assume that there is a subring $\Lambda \subset K$ which is of finite type over $\overline{\mathbb{Q}}$, and a scheme $\mathcal{X}$ which is proper over $S = \mathrm{Spec}(\Lambda)$ such that $\mathcal{X} \otimes_\Lambda K = X$ and that $\delta$ extends to a derivation $\xi$ of $\mathcal{O}$.

By generic smoothness of varieties over algebraically closed fields, we may assume after localization that $S = \mathrm{Spec}(\Lambda)$ is smooth over $\overline{\mathbb{Q}}$. Since $S$ is smooth, and the geometric generic fibre $X$ of $\mathcal{X}$ over $S$ is connected, we know that $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \Lambda$, and hence $\xi$ induces a derivation $\overline{\xi}$ of $\Lambda$ (which agrees with $\delta$ via the inclusion $\Lambda \subset K$). If $\overline{\xi}$ is trivial, then $\Lambda \subset K^\delta$ and we can take $F = K^\delta$.

If $\overline{\xi}$ is non-zero, then there is a closed point $x \in S$ at which $\overline{\xi}$ (which may now be viewed as a section of the tangent bundle of $S$ over $\overline{\mathbb{Q}}$) does not vanish. Since $S$ is smooth at $x$, there is a regular system of parameters $\{z_1, \ldots, z_n\}$ of $\mathcal{O}_{S,x}$, and since $\overline{\xi}$ does not vanish at $x$, there is at least one $i$ for which $\overline{\xi}(z_i)$ does not vanish at $x$. Without loss of generality, we may take $i = 0$. After further localization, we may assume that all the $z_i$ are regular on $S$, that the induced map $\pi = (z_1, \ldots, z_n) : S \to \mathbb{A}^d_{\overline{\mathbb{Q}}}$ is étale, and that $\overline{\xi}(z_1)$ vanishes nowhere on $S$.

Let $R$ be the discrete valuation ring which is the localization of $\Lambda$ at the prime ideal generated by $z_1$. We may then take $f = z_1$. Then the map $\pi : S \to \mathbb{A}^d_{\overline{\mathbb{Q}}}$ realizes $R$ as an étale extension of the discrete valuation ring $\overline{\mathbb{Q}}[z_1, z_2, \ldots, z_d]_{(z_1)}$, and the residue field $k$ of $R$ is a finite separable extension of $\overline{\mathbb{Q}}(z_2, \ldots, z_d)$ since $\pi$ is étale. Since $R$ contains $\overline{\mathbb{Q}}[z_2, \ldots, z_d]$ as a subring which injects into the residue field of $R$, it follows that there is an inclusion $\overline{\mathbb{Q}}(z_2, \ldots, z_d) \subset R$. Finally we set $E$ equal to the algebraic closure of $\overline{\mathbb{Q}}(z_2, \ldots, z_d)$ in $R$. □

3.2.4. *step 4.* Let $R^h$ be a strict Henselization of $R$. Then the algebraic closure of $E$ in $R^h$ maps isomorphically onto the residue field of $R^h$. Since the fraction field of $R^h$ is an algebraic extension of the fraction field of $R$, and $K$ is algebraically closed, there is an embedding $R^h \subset K$ extending $R \subset K$.

The isomorphism $\mathcal{X} \otimes_R K \simeq X$ extends uniquely to an isomorphism $\mathcal{X} \otimes_R R^h \otimes_{R^h} K \simeq X$. Since $R \subset R^h$ is formally étale, $\xi$ lifts uniquely to a derivation of the structure sheaf of $\mathcal{X} \otimes_R R^h$, and this lift is compatible with base change from $R^h$ to $K$. Note that $f \in R \subset R^h$ is also the generator of the maximal ideal of $R^h$.

**Lemma 20.** *Let $(A, \delta)$ be a differential ring, and $I = fA$ a principal ideal in $A$. For $n \geq 0$, write $A_n$ for $A/I^n$. Suppose that $\delta(f)$ is a unit mod $I$; then composition*

$$\phi : A \overset{\exp(\delta)}{\to} A[[t]] \to B[[t]]$$

*induces a compatible system of ring isomorphisms:*

$$A_n \to B[t]/(t^n)$$

*Proof.* First observe that the composition of $\phi$ with the map $B[[t]] \to B$ sending $B$ to 0 maps $I$ to 0, and hence $\phi(I) \subset tB[[t]]$. Let us write $J = tB[[t]]$. To prove the lemma it suffices to show that for all $k \geq 0$ the induced map $I^k/I^{k+1} \to J^k/J^{k+1}$ is an isomorphism for all $k \geq 0$. We start with the case $k = 0$, where it is clear that $A/I \simeq B \to B[[t]]/tB[[t]] \simeq B$ is an isomorphism. For $k \geq 1$, $I^k/I^{k+1}$ is a free rank one $B$-module generated by $f^k + I^{k+1}$ while $J^k/J^{k+1}$ is a free rank one $B$-module generated by $t^k + J^{k+1}$. Since $\phi(f) = \delta(f)t + O(t^2)$,with $\delta(f)$ a unit, $\phi(f^k) = (\delta(f))^k t^k + O(t^{k+1})$ is a generator of $J^k$ and it follows that $\phi$ induces an isomorphism $I^k/I^{k+1} \to J^k/J^{k+1}$ as desired. □

3.2.5. *Algebraization.* We may apply the lemma to $\mathcal{X}$ and the sheaf of ideals on $\mathcal{X}$ generated by $f$. The vector field $\xi$ on $\mathcal{X}$ induces maps

$$\phi_n : \mathcal{X}_n \to Y \otimes_{F=R/(f)} A[t]/(t^n) \qquad ,$$

where $Y$ is the fibre of $\mathcal{X}$ over the closed point of $\mathrm{Spec}(R)$, which by the lemma are isomorphisms. *I.e.*, $\xi$ induces an isomorphism of formal schemes

$$\widehat{\phi} = \exp(\xi) : \widehat{\mathcal{X}} \to Y \widehat{\otimes}_{F=R/(f)} \widehat{\mathbb{A}^1}$$

where $\widehat{\mathcal{X}}$ is the formal scheme which is the formal completion of $\mathcal{X}$ with respect to the sheaf of ideals generated by $f$. This isomorphism is an isomorphism of formal schemes over the formal scheme $\mathrm{Specf}(\widehat{R} \simeq F[[t]])$, and hence by [9], Ch. III, Th. 5.4.1, since $\mathcal{X}$ and $Y$ are both proper over $\widehat{R} \simeq F[[t]]$, $\exp(\xi)$ is algebraizable, *i.e.*, it is induced by an isomorphism of schemes (not just formal schemes!)

$$\phi : \mathcal{X} \otimes_R \widehat{R} \to Y \otimes_{F=R/(f)} (\widehat{R} \simeq F[[t]]).$$

3.2.6. *Approximation.* Now Artin approximation [1] implies, since $R$ is Henselian, that there exists an isomorphism

$$\phi' : \mathcal{X} \to Y \otimes_{F=R/(f)} R \qquad .$$

(Note that $\phi'$ may be chosen to agree with $\phi$ (and hence $\widehat{\phi}$) to any given finite order, but does not necessarily induce the map $\widehat{\phi}$.) Finally we observe that by taking $\otimes_R K$,we obtain an isomorphism

$$X = \mathcal{X} \otimes_R K \to Y \otimes_F K$$

and we are done.

3.3. **Remark.** The only place that we used that $X$ is proper was in the algebraization of isomorphisms in the category of formal schemes over $\mathrm{Specf}(F[[t]])$. Thus Buium's result will hold for any subcategory of the category of schemes with this property.

## 4. Complements and Questions

4.1. **Hasse-Schmidt Differentiation.** It is natural to ask to ask to what extent the methods used above to prove Buium's theorem's can be used in characteristic $p$.

Note that the proof of lemma 20 does not use that $\delta$ is a derivation, but only that $\phi$ is a ring homomorphism. In general a homomorphism from a ring $R$ to the ring $R[[t]]$ of formal power series over $R$ which, when composed with the augmentation $R[[t]] \to R$ which sends $t$ to 0, is identity, is known as a *Hasse-Schmidt differentiation.* If one writes such a differentiation as $r \mapsto \sum_i D^i(r)t^i$, then the $D^i$

are differential operators of order $i$. (See [11], §16 for the definition of a differential operator.) The fact that this is ring homomorphism implies that these operators satisfy the usual formula with respect to multiplication:

$$D^n(rs) = \sum_{i=0}^{n} D^i(r)D^{n-i}(s) \ .$$

In order for the differentiation to be a derivation, you additional information:

**Definition 6.** *A* Hasse-Schmidt differentiation *$\phi$ on a scheme $X$ is a map of schemes $\phi : X\widehat{\otimes}_{\mathbb{Z}}\mathbb{Z}[[t]] \to X$. Let $X$ be a scheme over a base $S$, and let $\mathcal{G}$ be a one parameter formal group over $S$, then a flow $\phi$ on a scheme $X$ over $S$ is said to be* $\mathcal{G}$-iterative, *with respect to $S$, if $\phi$ is an action of $\mathcal{G}$ on $X$. Note that in characteristic zero, all one parameter formal groups are isomorphic to the formal additive group $\mathcal{G}_a$, in which case a flow is iterative if and only if it is the integral of a vector field. See* [17].

In Lemma 20, we are given a flow $\phi$ on a scheme $p : \mathcal{X} \to S$, proper, and geometrically connected over the spectrum $S = \mathrm{Spec}(\lambda)$ of a discrete valuation ring, with the property that if $\xi$ is the associated vector field, and $\pi$ is the generator of the maximal ideal in $\Lambda$ then $\xi(\pi)$ is a unit modulo $\pi$. Of course once one no longer assumes that the flow is the exponential of the associated vector field, the flow is not determined by a finite amount of data - thus one cannot argue as in 3.2.3 that one can reduce to a field of finite transcendence degree over the prime field.

Notice that given a flow $\phi : A \to A[[t]]$, if we write the flow as:

$$\phi : a \mapsto \sum_i D_i(a)t^i$$

then the $D_i$ are differential operators. (Equal to $(D_1)^i/i!$ when the flow is the exponential of a vector field).

Thus one can ask:

**Question 1.** *Suppose that $X$ is a variety, projective and geometrically connected over a field $k$, possibly of characteristic greater than zero. Consider the algebra $\mathcal{D} = H^0(X, \mathcal{D}iff_k(\mathcal{O}_X, \mathcal{O}_X))$ of global sections of the sheaf $\mathcal{D}iff_k(\mathcal{O}_X, \mathcal{O}_X)$ of $k$-linear differential operators on $\mathcal{O}_X$. Then $\mathcal{D}$ acts on $H^0(X, \mathcal{O}_X) \simeq k$, and we ask whether $X$ is defined over the field of constants $k^{\mathcal{D}}$, or at least over its algebraic closure.*

### 4.2. Derivations and Valuation Rings.

See [5] chapter VI, [21], and [8] for valuation rings.

Recall that a valuation ring is a local domain $R$ such that if $x \in F$, $F$ being the fraction field of $R$, then if $x \notin R$, $1/x \in \mathfrak{m}$, $\mathfrak{m}$ being the maximal ideal of $R$. There is an associated valuation $v : F^* \to \Gamma$, where $\Gamma$ is the totally ordered abelian group $F^*/R^*$, with the order given by $v(x) \geq 0$ if and only if $x \in R$. Given a totally ordered abelian group $\Gamma$, a *valuation* $v$ on $F$ with values in $\Gamma$ is a homomorphism $v : F^* \to \Gamma$, such that $v(x + y) \geq \mathrm{Min}(v(x), v(y))$. Given such a valuation, the set $\{x \in F | v(x) \geq 0\}$ is a subring $R_v$, which is easily seen to be a valuation ring with maximal ideal $\mathfrak{m}_v = \{x \in F | v(x) > 0\}$, and the valuation induces an injection $F^*/R_v^* \subset \Gamma$. Generally when we speak of a valuation on a field we assume that this inclusion is the identity. Valuation rings are in general *not* noetherian, how they do still have some nice properties:

**Lemma 21.** *Let $R$ be a valuation ring. Then*

1. *Any finitely generated torsion free $R$-module is free.*
2. *Any torsion free $R$-module is flat.*

*Proof.* Exercise; see [5] Lemma 1 of VI.3.6.

□

**Question 2.** *Let $R$ be a valuation ring containing a differential field of characteristic zero. Is there a "simple" proof (or at least a proof that is not equivalent to proving uniformization) that $R^{\infty}$ is an integral domain?*

For example it would be enough to know that $\Omega_{R/k}$ is torsion free. (By Lemma 21 this would imply that it is a union of free submodules, $\Omega_{R/k} = \bigcup_{\alpha} F_{\alpha}$ and hence that $R^{\infty} = \varinjlim_{\alpha} \mathbb{S}^{*}_{R}(F_{\alpha})$ is a direct limit of integral domains, and hence is an integral domain.) However I am told by at least one of the experts that this statement is quite probably equivalent to uniformization.

Another remark is that since $R$ is a domain, we know by Kolchin's theorem that $R^{\infty}$ has a unique minimal ideal. Hence it would also suffice to show that $R^{\infty}$ is reduced.

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Department of Mathematics, Statistics, and Computer Science, (m/c 249), University of Illinois at Chicago,, 851 S. Morgan Street,, Chicago, IL 60607
*E-mail address*: `henri@math.uic.edu`