SO(3), SU(2), etc

$SL(n,\mathbb{R}), SL(n,\mathbb{C}) = \dots$

$SO(3) := \{A \in GL(3,\mathbb{R}) \mid A^tA = I\} = \{A : \mathbb{R}^3 \to \mathbb{R}^3 \text{ linear} \mid \|Ax\| = \|x\|\}$, $x \mapsto Ax$, $\det A = 1$

$\|x\|^2 = x^t x$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\|x\|^2 = x_1^2 + x_2^2 + x_3^2$

So $A^tA = I \underset{\forall x}{\iff} x^t A^t A x = x^t x$

$SU(2) = \{B \in GL(2,\mathbb{C}) \mid B^*B = I\}$, $B^* = \bar{B}^t$.

(T1) $\exists$ surj homo $SU(2) \to SO(3)$ w/ kernel $\{\pm I\}$.

(T2) $SU(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \,\middle|\, |z|^2 + |w|^2 = 1 \right\} (= S^3 \subset \mathbb{R}^4)$

(C) $\pi_1(SO(3)) = \mathbb{Z}_2$

Pf of T2 $B \in SU(2) \Rightarrow$

$\underbrace{\begin{pmatrix} \bar{z} & \bar{v} \\ \bar{w} & \bar{t} \end{pmatrix}}_{B^*} \underbrace{\begin{pmatrix} z & w \\ v & t \end{pmatrix}}_{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$zt = wv + 1$, $\bar{z}z + \bar{v}v = 1$, $\bar{z}w + \bar{v}t = 0$, $z\bar{w} + v\bar{t} = 0$, $\bar{w}w + \bar{t}t = 1$

elim z: $\bar{w}zt = w\bar{w}v + \bar{w}$, $z\bar{w}t = -v\bar{t}t$

$\Downarrow$

$v(w\bar{w} + \bar{t}t) = -\bar{w}$

$\Downarrow$

$z = \bar{t} \Leftarrow zt = \bar{t}t \Leftarrow zt = -w\bar{w} + 1 \Leftarrow v = -\bar{w}$

Pf of T1 Def $su(2) = \{M \in Mat_2(\mathbb{C}) \mid M + M^* = 0, \ tr\, M = 0\} = \left\{ \underbrace{\begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix}}_{=X} \,\middle|\, x_i \in \mathbb{R} \right\}$

$\|X\| = \det X$

$\| \quad \left\{ \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{=X} \,\middle|\, x_i \in \mathbb{R} \right\} = \mathbb{R}^3$

$SU(2) \xrightarrow{\varphi} SO(3)$

$B \longmapsto (X \mapsto BXB^*)$

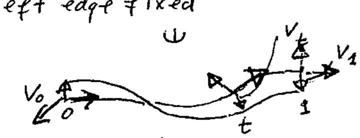
$\|BXB^*\| = \det(BXB^*) = \det X = \|X\|$.

$\operatorname{Ker}\varphi = \{B \mid BXB^* = X \ \forall X\} = \{B \mid BX = XB \ \forall X\} = \{B \in SU(2) \mid B \text{ scalar}\} = \{\pm I\}$

surj lăsată. $\square$.

Comment on C $\{$configurations of belt in $\mathbb{R}^3$ w/ left edge fixed$\} \to \{$paths $[0,1] \to SO(3)\}$

v_i = frame



$\longmapsto (t \mapsto R_t = \text{rot mapping } v_t' \text{ to } v_0')$ (v_t, v_0 moved to origin)

loops corr to config's w/ $v_1' = v_0'$

2 loops homotopic if config's obtained from one another by keeping v_1' fixed

Juxtaposition of config corresponds to compos of paths,

So $\forall$ config juxt. with itself can be disentangled by keeping v_0 fixed & $v_1' = v_0'$.

e.g. belt twisted twice

(Demonstration w/ my belt)

Examples of Lie groups in physics

$GL(K^n)$
$\parallel$

$M(n,\mathbb{C})$ not a group.

$GL(n,\mathbb{C})$, $GL(n,\mathbb{R})$, $K=\mathbb{R},\mathbb{C}$, $GL(V) = \cdots$, $GL(n,K)$

$SL(n,K) = \{A \in GL(n,K) \mid \det A = 1\}$, $A^* = \bar{A}^t$

$U(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid AA^* = I\}$

$SU(n,\mathbb{C}) = U(n,\mathbb{C}) \cap SL(n,\mathbb{C})$; $SU(2,\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \;\middle|\; |a|^2+|b|^2=1 \right\}$

$S^3 \subset \mathbb{R}^4$, $x_1^2+\cdots+x_4^2 = x^t x$

$O(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid AA^t = I\} = \{A \in GL(n,\mathbb{R}) \mid \|Ax\|^2 = \|x\|^2,\ x \in \mathbb{R}^n\}$

$Q = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$, $L = \{A \in GL(4,\mathbb{R}) \mid A^t Q A = Q\}$

$q(x) = x^t Q x = x_0^2 - x_1^2 - x_2^2 - x_3^2$ $\Rightarrow$ $L = \{A \in \underbrace{GL(4,\mathbb{R})}_{GL(\mathbb{R}^4)} \mid q(Ax) = q(x),\ x \in \mathbb{R}^4\}$ = "aut of Mink space"

$(\mathbb{R}^4, q)$ Minkovski space

$H(2,\mathbb{C}) = \{A \in M(2,\mathbb{C}) \mid A^* = A\}$ not a group

$\mathbb{R}^4 \xrightarrow{\sim} H(2,\mathbb{C})$, $(x_0,x_1,x_2,x_3) \mapsto \begin{pmatrix} x_0+x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}$. view as identify.

$\det(x) = q(x)$.

$\pi : SL(2,\mathbb{C}) \to GL(H(2,\mathbb{C})) \simeq GL(\mathbb{R}^4)$

$A \longmapsto \underline{A}$, $\underline{A}(B) = ABA^*$ well def b/c $(ABA^*)^* = ABA^*$

• $\operatorname{Im}\pi \subset L$ b/c

$q(\underline{A}(B)) = \det(\underline{A}(B)) = \det(ABA^*) = \det(B) = q(B)$

so $\boxed{SL(2,\mathbb{C}) \to L}$

• $\pi(SU(2,\mathbb{C})) \subset \{T \in L \subset GL(a(4,\mathbb{C})) \mid T(I) = I\} \simeq O(3,\mathbb{R})$

b/c if $A \in SU(2,\mathbb{C}) \Rightarrow \underline{A}(I) = AIA^* = AA^* = I$

b/c LHS is $\simeq \{T \in L \subset GL(\mathbb{R}^4) \mid Te_1 = e_1\}$ $\simeq O(3,\mathbb{R})$

b/c T fixes $e_1 \Rightarrow T$ fixes orth compl of e_1 etc. w/t $\langle x,y\rangle = x^t Q y$

$S^3 \simeq \boxed{SU(2,\mathbb{C}) \to SO(3,\mathbb{R})}$ fact: surj w/ kernel $\mathbb{Z}_2$

S^1 Lie gr, S^2 not a Lie gr, S^3 Lie gr, ...?