Arithmetic differential geometry

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This talk is partly based on joint papers with

Taylor Dupuy,
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and on the monograph

A. Buium, Foundations of arithmetic differential geometry, Math Surveys and Monographs. AMS, 2017.

The theory behind this talk was developed partly in joint papers with

James Borger
Yuri I. Manin
Emma Previato
Arnab Saha
Santiago Simanca

Plan of the talk:

Part I: Outline of the theory

Part II: Comparison with other theories

Part III: Open problems

Part I: Outline of the theory

• STARTING POINT:
A.B., "Differential characters of abelian varieties over p-adic fields", Inventiones 1995.

Idea there: Fermat quotients instead of derivations used to construct arithmetic analogues of the Lie-Cartan jet spaces.

Subsequently: latter used in 2 contexts:

a) arithmetic on Abelian and Shimura varieties
b) arithmetic on classical groups $GL_n, SO_n, Sp_n, ...$

• For a) see book "Arithmetic differential equations" AMS 2005; one is led to arithmetic analogues of Picard-Fuchs, Manin kernels, modular forms, etc.

• For b) book "Foundations of arithmetic differential geometry", AMS 2017; one is led to arithmetic analogues of Riemannian and symplectic geometry.
THIS TALK IS ABOUT b).

In this talk:

- $\mathbb{Z}$ analogous to a ring of functions on $M = \mathbb{R}^m$, $m \to \infty$
- primes $p \in \mathbb{Z}$ analogous to coordinate functions $\xi_i : M \to \mathbb{R}$, $i \leq m$
- $\frac{\partial f}{\partial \xi_i}$ replaced by Fermat quotients $\frac{n - n^p}{p}$
- metrics replaced by symmetric integral matrices
- connections replaced by adelic objects attached to such matrices
- curvature replaced by global objects attached to such matrices

Classical versus arithmetic differentiation

- $A \subset C^{\infty}(\mathbb{R}^m, \mathbb{R})$
- $A = \mathbb{Z}[1/M, \zeta_N]$
- $\mathcal{U} = \{\xi_1, \xi_2, \ldots, \xi_m\} \subset C^{\infty}(\mathbb{R}^m, \mathbb{R})$
- $\mathcal{V} = \{p_1, p_2, p_3, \ldots\} \subset \mathbb{Z}, \quad m = |\mathcal{V}| \leq \infty$

- For $A \subset C^{\infty}(\mathbb{R}^m, \mathbb{R})$,

$$\delta_i : A \to A, \quad \delta_i f := \frac{\partial f}{\partial \xi_i}, \quad , i \in \{1, ..., m\}. \tag{1}$$

- For $A = \mathbb{Z}[1/M, \zeta_N]$

$$\delta_p : A \to A, \quad \delta_p(a) = \frac{\phi_p(a) - a^p}{p}, \quad p \in \mathcal{V}, \tag{2}$$

where $\phi_p : A \to A$ the unique ring automorphism sending ζ_N into ζ_N^p.

• A *derivation* on a ring B is an additive map $B \to B$ that satisfies the Leibniz rule.

• A *p-derivation* defined as follows. Assume B is a ring and assume, for simplicity, that p is a non-zero divisor in B; then a *p-derivation* on B is a set theoretic map

$$\delta_p : B \to B$$

with the property that the map

$$\phi_p : B \to B$$

defined by

$$\phi_p(b) = b^p + p\delta_p b$$

is a ring homomorphism.

• We denote by ϕ_p the ring homomorphism attached to a p-derivation δ_p and we shall refer to ϕ_p as the *Frobenius lift* attached to δ_p.

Classical differential geometry revisited

- $n \times n$ matrix $x = (x_{ij})$ of indeterminates
- For $A \subset C^{\infty}(\mathbb{R}^m, \mathbb{R})$,

$$B = A[x, \det(x)^{-1}]. \tag{3}$$

- A *connection* on $P = \mathbb{R}^m \times GL_n$ is a tuple $\delta = (\delta_i)$ of derivations

$$\delta_i : B \to B, \quad i \in \{1, ..., m\}, \tag{4}$$

extending the derivations 1.

- For δ define the *curvature* as the matrix (φ_{ij}) of commutators

$$\varphi_{ij} := [\delta_i, \delta_j] : B \to B, \quad i, j \in \{1, ..., m\}. \tag{5}$$

- The *holonomy ring* of δ is the $\mathbb{Z}$-span $\mathfrak{hol}$ in

$$\mathrm{End}(B) := \mathrm{End}_{\mathbb{Z}-mod}(B)$$

of the commutators

$$[\delta_{i_1}, [\delta_{i_2}, [...., [\delta_{i_{s-1}}, \delta_{i_s}]...]]], \tag{6}$$

where $s \geq 2$; it is a Lie subring of $\mathrm{End}(B)$.

• The *trivial connection* $\delta_0 = (\delta_{0i})$ is defined by

$$\delta_{0i} x = 0.$$

• A connection is *flat* if its curvature vanishes: $\varphi_{ij} = 0$ for all $i, j = 1, ..., m$. For instance δ_0 is flat.

Types of connections for which we seek arithmetic analogues:

1) Ehresmann connections,
2) Chern connections,
3) Levi-Civita connections,
4) Fedosov connections,
5) Lax connections,
6) Hamiltonian connections,
7) Cartan connections.

Will concentrate on 1) through 5).

• A connection (δ_i) is called an *Ehresmann connection* if it satisfies one of the following two equivalent conditions:

1a) There exist $n \times n$ matrices A_i with coefficients in $A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$ such that

$$\delta_i x = A_i x \tag{7}$$

1b) The following diagrams are commutative:

$$\begin{array}{ccc} B & \xrightarrow{\mu} & B \otimes_A B \\ \delta_i \downarrow & & \downarrow \delta_i \otimes 1 + 1 \otimes \delta_{0i} \\ B & \xrightarrow{\mu} & B \otimes B \end{array} \tag{8}$$

• Condition 1a: (δ_i) is *right linear*.

• Condition 1b: (δ_i) is *right invariant*.

• The arithmetic analogues of conditions 1a and 1b cease to be equivalent.

• If (δ_i) is an Ehresmann connection the curvature satisfies $\varphi_{ij}(x) = F_{ij}x$ where F_{ij} is the matrix given by the classical formula

$$F_{ij} := \delta_i A_j - \delta_j A_i - [A_i, A_j]. \tag{9}$$

• There is a Galois theory attached to flat Ehresmann connections, the Picard-Vessiot theory.

• Indeed, for a flat Ehresmann connection $\delta = (\delta_i)$ consider the *logarithmic derivative map* $l\delta : GL_n(A) \to \mathfrak{gl}_n(A)^m$ with coordinates

$$l\delta_i(u) = \delta_i u \cdot u^{-1},$$

where $\mathfrak{gl}_n$ is the Lie algebra of GL_n.

• The fibers of the map $l\delta$ are solution sets of systems of linear equations

$$\delta_i u = A_i \cdot u.$$

• If one replaces A by a ring of complex analytic functions then Galois groups can be classically attached to such systems; these groups are algebraic subgroups of $GL_n(\mathbb{C})$ measuring the algebraic relations among the solutions to the corresponding systems.

• We next discuss ("real") Chern connections.

• In classical differential geometry Chern connections form a subclass of Ehresmann connections. This will not be the case in arithmetic differential geometry.

• For $A \subset C^{\infty}(\mathbb{R}^m, \mathbb{R})$ let $q \in GL_n(A)$ with $q^t = \pm q$; think: metric/2-form.

• For $G = GL_n = Spec\ B$ consider the maps of schemes over A,

$$\mathcal{H}_q : G \to G, \quad \mathcal{B}_q : G \times G \to G \tag{10}$$

defined by

$$\mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x, y) = x^t q y. \tag{11}$$

• Recall the *trivial* connection $\delta_0 = (\delta_{0i})$ on G defined by $\delta_{0i}x = 0$.

• Non-standard reformulation of a classical fact: there is a unique connection (δ_i) on G such that the following diagrams are commutative:

$$\begin{array}{ccc} B & \stackrel{\delta_i}{\longleftarrow} & B \\ \mathcal{H}_q \uparrow & & \uparrow \mathcal{H}_q \\ B & \stackrel{\delta_{0i}}{\longleftarrow} & B \end{array} \qquad \begin{array}{ccc} B & \stackrel{\delta_i \otimes 1 + 1 \otimes \delta_{0i}}{\longleftarrow} & B \otimes_A B \\ \delta_{0i} \otimes 1 + 1 \otimes \delta_i \uparrow & & \uparrow \mathcal{B}_q \\ B \otimes_A B & \stackrel{\mathcal{B}_q}{\longleftarrow} & B \end{array} \qquad (12)$$

• This δ turns out to be Ehresmann and can be referred to as the *Chern connection* attached to q.

- To see the analogy with the classical Chern connection set

$$\Gamma^k_{ij} = -A_{ikj} = (k,j) \text{ entry of } (-A_i := -\delta_i x \cdot x^{-1}).$$

(Christoffel symbols of the 2nd kind)

$$\Gamma_{ijk} := \Gamma^l_{ij} q_{lk},$$

(Christoffel symbols of the 1st kind).

- The commutativity of the left diagram in 12 is equivalent to the condition

$$\delta_i q_{jk} = \Gamma_{ijk} \pm \Gamma_{ikj}, \tag{13}$$

- The commutativity of the right diagram in 12 is equivalent to the condition

$$\Gamma_{ijk} = \pm\Gamma_{ikj}; \tag{14}$$

- So the Chern connection attached to q exists and is unique, given by

$$\Gamma_{ijk} = \frac{1}{2}\delta_i q_{jk}. \tag{15}$$

• We next discuss Levi-Civita connections.

• Again, in classical differential geometry, these are special cases of Ehresmann connections (which will not be the case in arithmetic differential geometry).

• Assume here that $n = m$. (Note we also implicitly assume here that a bijection is given between the set indexing the derivations δ_i and the set indexing the rows and columns of the matrix x_{ij}; such a bijection plays the role of what is classically called a *soldering*.)

• The connection (δ_i) is *torsion free* if:

$$\Gamma_{ijk} = \Gamma_{jik}. \tag{16}$$

• The fundamental theorem of Riemannian geometry says: for $q^t = q$, there is a unique connection δ satisfying the equations 13 (with the $+$ sign) and 16; it is given by

$$\Gamma_{kij} = \frac{1}{2}\left(\delta_k q_{ij} + \delta_i q_{jk} - \delta_j q_{ki}\right) \tag{17}$$

• It is Ehresmann and it is called the *Levi-Civita* connection attached to q.

• The Levi-Civita connection is generally different from the Chern connection; it coincides with the Chern connection if and only if

$$\delta_i q_{jk} = \delta_j q_{ik} \tag{18}$$

in which case q is called *Hessian* (the "real" analogue of *Kähler*).

• For the Levi-Cività connection attached to a metric $q = (q_{ij})$ we set:

$$\varphi_{ij}(x) = F_{ij}x, \quad F_{ij} := \delta_i A_j - \delta_j A_i - [A_i, A_j], \tag{19}$$

$$F_{ij} = (F_{ijkl}), \quad R^k_{lij} := -F_{ijkl}, \quad R_{ijkl} = q_{im} R^m_{jkl}. \tag{20}$$

One refers to R_{ijkl} as the *covariant Riemann tensor*; the latter has following symmetries:

$$R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} + R_{iklj} + R_{ijlk} = 0, \quad R_{ijkl} = R_{klij}. \tag{21}$$

• A metric q is said to have *constant sectional curvature* if there exists $\kappa \in A$ with $\delta_i \kappa = 0$ such that

$$R_{ijkl} = \kappa \cdot (q_{ik} q_{jl} - q_{jk} q_{il}). \tag{22}$$

• If a metric q is *normal* at the origin 0 (i.e. $q \equiv 1 \mod M^2$ where 1 is the identity matrix and $M \subset A$ is the ideal of functions vanishing at 0) then the covariant Riemann tensor satisfies:

$$R_{ijkl} \equiv \frac{1}{2}(\delta_j\delta_k q_{il} + \delta_i\delta_l q_{jk} - \delta_i\delta_k q_{jl} - \delta_j\delta_l q_{ik}) \mod M. \tag{23}$$

• A "symplectic analogue" of Levi-Civita connections are *Fedosov* connections. They are not necessarily Ehresmann connections; and the Ehresmann ones are not unique.

• $q \in GL_n(A)$, $q^t = -q$, so n is even. A *Fedosov connection* relative to q is an invariant connection δ that is torsion free (i.e. 16 holds) and satisfies 13 (with the $-$ sign).

• A Fedosov connection relative to q exists if and only if q is *symplectic* in the sense that it satisfies

$$\delta_i q_{jk} + \delta_j q_{ki} + \delta_k q_{ij} = 0. \tag{24}$$

• For a given symplectic q Ehresmann Fedosov connections exist and are not unique; one such connection is given by

$$\Gamma_{ijk} = \frac{1}{3}\left(\delta_i q_{jk} + \delta_j q_{ik}\right). \tag{25}$$

• Finally we discuss *Lax connections*; unlike Chern and Levi-Civita connections, Lax connections are *not* a subclass of the Ehresmann connections.

• (δ_i) is Lax if

$$\delta_i x = [A_i(x), x] := A_i(x)x - xA_i(x) \tag{26}$$

for some $n \times n$ matrix $A_i(x)$ with coefficients in B.

• If $A_i(x) = A_i$ have coefficients in A (independent of x) and F_{ij} is the curvature of the connection $x \mapsto A_i x$ then the curvature of Lax connection is given by

$$\varphi_{ij}(x) = [F_{ij}, x] \tag{27}$$

so it completely determines the trace free part of (F_{ij}).

- For Lax connections the following diagrams are commutative:

$$\begin{array}{ccc} B & \stackrel{\delta_i}{\longleftarrow} & B \\ \mathcal{P}\uparrow & & \uparrow\mathcal{P} \\ A[z] & \stackrel{\delta_{0i}}{\longleftarrow} & A[z] \end{array} \tag{28}$$

where $A[z] = A[z_1, ..., z_n]$ is a ring of polynomials in the variables z_j, δ_{0i} are the unique derivations extending the corresponding derivations on A with $\delta_{0i} z_j = 0$, and $\mathcal{P}$ is the A-algebra homomorphism with $\mathcal{P}(z_j) = \mathcal{P}_j(x)$,

$$\det(s \cdot 1 - x) = \sum_{j=0}^{n} (-1)^j \mathcal{P}_j(x) s^{n-j}.$$

• The commutativity of 28 expresses the fact that the Lax connections describe "isospectral flows" on GL_n: if u is a solution to

$$\delta_i u = [A_i(u), u]$$

then

$$\delta_i(\mathcal{P}_j(u)) = 0$$

so the eigenvalues of u are δ_i-constant.

• The real theory above has a complex analogue (and hence a "(1, 1)-analogue")

• $M = \mathbb{C}^m$, $A \subset C^\infty(M, \mathbb{C})$ stable under

$$\delta_i := \frac{\partial}{\partial z_i}, \quad \delta_{\bar{i}} := \frac{\partial}{\partial \overline{z}_i}, \quad i = 1, ..., m,$$

• A connection on $G = Spec\ B$, $B = A[x, \det(x)^{-1}]$, is an m-tuple of derivations $\delta_i : B \to B$ extending the derivations $\delta_i : A \to A$.

• For $\delta_{\bar{i}} : B \to B$, extending the derivations $\delta_{\bar{i}} : A \to A$, such that $\delta_{\bar{i}} x = 0$ define the $(1, 1)$-curvature of $\delta = (\delta_i)$ as the matrix $(\varphi_{i\bar{j}})$ with entries the A-derivations

$$\varphi_{i\bar{j}} := [\delta_i, \delta_{\bar{j}}] : B \to B. \tag{29}$$

The theory proceeds from here.

Arithmetic differential geometry

• We introduce an arithmetic analogue of connection and curvature. We start with the analogue of the real case.

• The first step is clear: we consider a ring

$$B = A[x, \det(x)^{-1}]$$

with $A = \mathbb{Z}[1/M, \zeta_N]$.

• A first attempt to define arithmetic analogues of connections would be to consider families of p-derivations $\delta_p : B \to B$, $p \in \mathcal{V}$, extending the p-derivations 2; one would then proceed by considering their commutators on B (or, if necessary, expressions derived from these commutators). But the point is that the examples of "arithmetic analogues of connections" we will encounter in practice will almost never lead to p-derivations $B \to B$!

• What we shall be led to is, rather, an adelic concept we next introduce. (Our guiding "principle" here is that C^∞ geometric objects should correspond to adelic objects in arithmetic while analytic/algebraic geometric objects correspond to global objects in arithmetic.)

• For each $p \in \mathcal{V}$ we consider the p-adic completion of B:

$$B^{\widehat{p}} := \lim_{\leftarrow} B/p^n B. \tag{30}$$

• Define an *adelic connection* on $G = GL_n$ to be a family (δ_p) of p-derivations

$$\delta_p : B^{\widehat{p}} \to B^{\widehat{p}}, \quad p \in \mathcal{V}, \tag{31}$$

extending the p-derivations in 2.

• If $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ are the Frobenius lifts attached to δ_p and $G^{\widehat{p}} = Spf\ B^{\widehat{p}}$ is the p-adic completion of $G = GL_n = Spec\ B$ then we still denote by $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ the induced morphisms of p-adic formal schemes.

• Next: analogues of the various types of connections encountered in classical differential geometry: Ehresmann, Chern, Levi-Civita, Fedosov, and Lax.

• Analogue of the trivial connection (δ_{0i}), $\delta_{0i}x = 0$ is the adelic connection (δ_{0p}) defined by $\delta_{0p}x = 0$. The associated Frobenius lifts (ϕ_{0p}) satisfy $\phi_{0p}(x) = x^{(p)}$ where $x^{(p)}$ is the matrix (x_{ij}^p).

• To introduce arithmetic analogues of Ehresmann connections one starts by noting that, for $n \geq 2$, there are no adelic connections (δ_p) whose attached Frobenius lifts (ϕ_p) make the following diagrams commute:

$$\begin{array}{rcl} G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \end{array} \tag{32}$$

• The above is an elementary observation; one can prove a less elementary result that for $n \geq 2$ and $p \not| n$ there are no adelic connections whose attached Frobenius lifts (ϕ_p) and (ϕ_{1p}) make the following diagrams commute:

$$\begin{array}{ccc} G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \\ \phi_p \times \phi_{1p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \end{array} \qquad (33)$$

• There is a useful property, weaker than the commutativity of 32, namely an invariance property with respect to the action of N on G by right translation, where

$$N \text{ is the normalizer of the diagonal maximal torus } T \text{ of } G. \qquad (34)$$

Indeed we will say that an adelic connection (δ_p) with associated Frobenius lifts (ϕ_p) is *right invariant with respect to N* if the following diagrams are commutative:

$$\begin{array}{rcl} G^{\widehat{p}} \times N^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times N^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \end{array} \qquad (35)$$

• This latter property has its own merits but is too weak to function appropriately as a defining property of Ehresmann connections in arithmetic. Instead, we can consider an appropriate analogue of "linear," 7. What we can do is replace the Lie algebra $\mathfrak{gl}_n$ by an arithmetic analogue of it, $\mathfrak{gl}_{n,\delta_p}$, and then we can introduce an arithmetic analogue of the logarithmic derivative.

• This new framework leads to the following definition: an adelic connection (δ_p) is an *Ehresmann connection* if

$$\delta_p x = \alpha_p \cdot x^{(p)}, \tag{36}$$

where α_p are matrices with coefficients in A. Ehresmann connections are right invariant with respect to N.

• One can attach Galois groups to such Ehresmann connections and develop the first steps of their theory. A natural expectation is that these Galois groups belong to the group $N(A)^{\delta}$ of all matrices in $N(A)$ whose entries are roots of unity or 0. This expectation is not always realized but one can prove (B-Dupuy) that something close to it is realized for (α_p) "sufficiently general."

• The above expectation is justified by the fact that, according to the general philosophy of the field with one element $\mathbb{F}_1$, the union of the $N(A)^{\delta}$'s, as A varies, plays the role of "$GL_n(\mathbb{F}_1^a)$," where $\mathbb{F}_1^a$ is the "algebraic closure of $\mathbb{F}_1$.

• Next: arithmetic analogue of Chern connections. Unlike in the case of classical differential geometry our adelic Chern connections will not be special cases of Ehresmann connections (although they *will* be right invariant with respect to N).

• Let $q \in GL_n(A)$ with $q^t = \pm q$. Attached to q we have, again, maps $\mathcal{H}_q : G \to G$ and $\mathcal{B}_q : G \times G \to G$ defined by $\mathcal{H}_q(x) = x^t qx$ and $\mathcal{B}_q(x, y) = x^t qy$. We continue to denote by $\mathcal{H}_q, \mathcal{B}_q$ the maps induced on the p-adic completions $G^{\widehat{p}}$ and $G^{\widehat{p}} \times G^{\widehat{p}}$. Consider the unique adelic connection $\delta_0 = (\delta_{0p})$ on G with $\delta_{0p}x = 0$ and denote by (ϕ_{0p}) the attached Frobenius lifts. Also, for an arbitrary adelic connection $\delta = (\delta_p)$ on G, denote by (ϕ_p) the Frobenius lifts attached to δ.

• Then one proves that there exists a unique adelic connection δ such that the following diagrams are commutative:

$$\begin{array}{ccc} G^{\widehat{p}} & \stackrel{\phi_p}{\longrightarrow} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \stackrel{\phi_{0p}}{\longrightarrow} & G^{\widehat{p}} \end{array} \qquad \begin{array}{ccc} G^{\widehat{p}} & \stackrel{\phi_{0p} \times \phi_p}{\longrightarrow} & G^{\widehat{p}} \times G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \mathcal{B}_q \\ G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mathcal{B}_q}{\longrightarrow} & G^{\widehat{p}} \end{array} \qquad (37)$$

• The adelic connection δ is referred to as the *Chern connection* (on $G = GL_n$) attached to q.

• Note the following relation between the "Christoffel symbols" defining our Chern connection and the Legendre symbol. Let $q \in GL_1(A) = A^\times$, $A = \mathbb{Z}[1/M]$, and let $\delta = (\delta_p)$ be the Chern connection associated to q. Then it turns out that $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ is defined by $\phi_p : \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}} \to \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}}$,

$$\phi_p(x) = q^{(p-1)/2} \left(\frac{q}{p} \right) x^p, \tag{38}$$

where $\left(\frac{q}{p} \right)$ is the Legendre symbol of $q \in A^\times \subset \mathbb{Z}_{(p)}$.

• Next: analogues of Levi-Civita connections. They are already relevant in case $\mathcal{V}$ consists of one prime p only.

• Analogue of fundamental theorem of Riemannian geometry: for $q \in GL_n(A)$, $q^t = q$, there is a unique n-tuple of $(\delta_{1p}, ..., \delta_{np})$ of adelic connections on $G = GL_n$, called the *Levi-Civita connection* attached to q, with attached Frobenius lifts $(\phi_{1p}, ..., \phi_{np})$, such that the following diagrams are commutative for $i = 1, ..., n$,

$$\begin{array}{rcl} G^{\widehat{p}} & \stackrel{\phi_{ip}}{\longrightarrow} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \stackrel{\phi_{0p}}{\longrightarrow} & G^{\widehat{p}} \end{array} \quad (39)$$

and such that, for all $i, j = 1, ..., n$, we have:

$$\delta_{ip} x_{kj} = \delta_{jp} x_{ki}. \quad (40)$$

• The conditions 39 and 40 are analogous to the conditions of parallelism 13 (with the $+$ sign) and torsion freeness 16 defining classical Levi-Civita connections.

• The Levi-Civita connection and the Chern connection attached to q are related by certain congruences mod p that are reminiscent of the relation between the two connections in classical differential geometry.

• The fact that in the Levi-Civita picture we restrict to the case of one prime only suggests that our picture corresponds to the classical Levi-Civita connections of *cohomogeneity one* metrics, by which we understand here metrics $g = \sum g_{ij} d\xi_i d\xi_j$ satisfying

$$\delta_k g_{ij} = \delta_l g_{ij}. \tag{41}$$

• One can also attempt to develop an arithmetic analogue of Fedosov connections as follows.

• For $q \in GL_n(A)$, $q^t = -q$ say that an n-tuple of $(\delta_{1p}, ..., \delta_{np})$ of adelic connections on $G = GL_n$ is a *Fedosov connection* relative to q if the attached Frobenius lifts $(\phi_{1p}, ..., \phi_{np})$ make the diagrams 39 commutative and, in addition the equalities 40 hold.

• One can prove that for $n = 2$ and any antisymmetric q Fedosov connections relative to q exist. However, in contrast with the Levi-Civita story, for $n \geq 4$ there is no Fedosov connection relative to the split q, for instance.

• Next: arithmetic analogues of Lax connections. In fact there are two such analogues which we call *isospectral* and *isocharacteristic* Lax connections.

• By the way isospectral and isocharacteristic Lax connections are not defined on the whole of G but rather on certain Zariski open sets

$$G^*, \quad G^{**} \subset G$$

respectively.

• If $G^* \subset G$ is the open set of regular matrices, T is the diagonal maximal torus and $T^* = T \cap G^*$ then there is an adelic connection, referred to as the *canonical isospectral Lax connection* that makes commutative the following diagrams:

$$\begin{array}{ccc} (T^*)^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\phi_{0p} \times \phi_{0p}}{\longrightarrow} & (T^*)^{\widehat{p}} \times G^{\widehat{p}} \\ \mathcal{C} \downarrow & & \downarrow \mathcal{C} \\ (G^*)^{\widehat{p}} & \stackrel{\phi_p}{\longrightarrow} & (G^*)^{\widehat{p}} \end{array} \qquad (42)$$

with $G^* \cap T = T^*$ and $\mathcal{C}(t, x) = x^{-1} t x$.

• More generally, if (δ_p) is the canonical isospectral Lax connection, with Frobenius lifts (ϕ_p), $\phi_p(x) = \Phi_p(x)$, and $\alpha_p(x)$ are $n \times n$ matrices with entries in $\mathcal{O}(G^*)^{\widehat{p}}$, then, setting

$$\epsilon_p(x) = 1 + p\alpha_p(x)$$

one can consider the *isospectral Lax connection* attached to (α_p), defined by the family of Frobenius lifts $(\phi_p^{(\epsilon)})$,

$$\phi_p^{(\epsilon)}(x) = \Phi_p^{(\epsilon)}(x) = \epsilon_p(x) \cdot \Phi_p(x) \cdot \epsilon_p(x)^{-1}.$$

The latter has the following property that justifies the term *isospectral*.

• Let

$$u_p = a_p^{-1} b_p a_p, \quad a_p \in GL_n(A^{\widehat{p}}), \quad b_p = diag(b_{p1}, ..., b_{pn}) \in T^*(A^{\widehat{p}})$$

be such that $\phi_p^{(\epsilon)}(u) = \Phi_p^{(\epsilon)}(u)$. Then

$$\delta_p b_{pi} = 0, \quad i = 1, ..., n.$$

So the eigenvalues b_{pi} of u_p are δ_p-constant.

• On the other hand we can prove: there exist adelic connections (δ_p) that make commutative the following diagrams:

$$\begin{array}{ccc} (G^{**})^{\widehat{p}} & \xrightarrow{\phi_p} & (G^{**})^{\widehat{p}} \\ \mathcal{P}\downarrow & & \downarrow\mathcal{P} \\ (\mathbb{A}^n)^{\widehat{p}} & \xrightarrow{\phi_{0p}} & (\mathbb{A}^n)^{\widehat{p}}, \end{array} \tag{43}$$

where $\mathbb{A}^n = Spec\ A[z]$. These diagrams are analogous to 28; the connections above are called isocharacteristic Lax connections. Among isocharacteristic Lax connections there is a *canonical* one.

• For any isocharacteristic Lax connection (δ_p), if $\phi_p(x) = \Phi_p(x)$ and $u_p \in G^{**}(A^{\widehat{p}})$ satisfies $\phi_p(u) = \Phi_p(u)$ we have that

$$\delta_p(\mathcal{P}_j(u)) = 0, \quad j = 1, ..., n.$$

So the coefficients of the characteristic polynomial of u_p are δ_p-constant.

- We encountered 2 conditions for a matrix u

1) The eigenvalues of u are δ_p-constant;

2) The coefficients of the characteristic polynomial of u are δ_p-constant.

- Recall: δ_p-constant implies being a root of unity or 0.
- So (unlike in classical calculus) 1) and 2) are not equivalent!!!
- So "isospectral" and "isocharacteristic" Lax connections are quite different!!!

• Next: curvature of adelic connections.

• Case of Ehresmann connections, 36. Since $\alpha_p \in A$ for all p our Frobenius lifts $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ induce Frobenius lifts $\phi_p : A[x] \to A[x]$ and hence one can consider the "divided" commutators

$$\varphi_{pp'} := \frac{1}{pp'}[\phi_p, \phi_{p'}] : A[x] \to A[x], \quad p, p' \in \mathcal{V}. \tag{44}$$

The family $(\varphi_{pp'})$ will be referred to as the *curvature* of the adelic connection (δ_p).

• The situation for general adelic connections (in particular for Chern and Lax connections) will be quite different. Indeed, in defining curvature we face the following dilemma: our p-derivations δ_p in 31 do not act on the same ring, so there is no a priori way of considering their commutators and, hence, it does not seem possible to define, in this way, the notion of curvature.

• It will turn out, however, that some of our adelic connections will satisfy a remarkable property which we call *globality along the identity* (more generally along various subvarieties) and which will allow us to define curvature via commutators.

• Consider the matrix $T = x - 1$, where 1 is the identity matrix. We say that an adelic connection (δ_p) on GL_n, with attached family of Frobenius lifts (ϕ_p), is *global along* 1 if, for all p, $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ sends the ideal of 1 into itself and, moreover, the induced homomorphism $\phi_p : A^{\widehat{p}}[[T]] \to A^{\widehat{p}}[[T]]$ sends the ring $A[[T]]$ into itself.

• If the above holds then one can consider the *curvature* of (δ_p) as the family of "divided" commutators $(\varphi_{pp'})$,

$$\varphi_{pp'} := \frac{1}{pp'}[\phi_p, \phi_{p'}] : A[[T]] \to A[[T]], \tag{45}$$

where $p, p' \in \mathcal{V}$. Call this procedure *analytic continuation between primes*.

• Define the *holonomy ring* $\mathfrak{hol}$ of δ as the $\mathbb{Z}$-linear span in $\mathrm{End}(A[[T]])$ of all the *Lie monomials*

$$[\phi_{p_1}, [\phi_{p_2}, ..., [\phi_{p_{s-1}}, \phi_{p_s}]...]] : A[[T]] \to A[[T]]$$

where $s \geq 2$, $p_i \in \mathcal{V}$.

• Define the *holonomy* $\mathbb{Q}$*-algebra* $\mathfrak{hol}_{\mathbb{Q}}$ of δ as the $\mathbb{Q}$-linear span of $\mathfrak{hol}$ in $\mathrm{End}(A[[T]]) \otimes \mathbb{Q}$.

• Define the *completed holonomy ring*,

$$\widehat{\mathfrak{hol}} = \lim_{\leftarrow} \mathfrak{hol}_n,$$

where $\mathfrak{hol}_n$ is the image of the map

$$\mathfrak{hol} \to \mathrm{End}(A[[T]]/(T)^n). \tag{46}$$

• The trivial adelic connection $\delta_0 = (\delta_{0p})$, $\delta_{0p}x = 0$, is global along 1 so it induces ring endomorphisms $\phi_{0p} : A[[T]] \to A[[T]]$,

$$\phi_{0p}(T) = (1 + T)^{(p)} - 1.$$

We may morally view δ_0 as an analogue of a flat connection in real geometry (where $A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$). Alternatively we may view δ_0 as an arithmetic analogue of the derivations $\delta_{\bar{i}} = \partial/\partial\bar{z}_i$ on $A[x, \det(x)^{-1}]$ which kill x, where $A \subset C^\infty(\mathbb{C}^m, \mathbb{C})$.

• Following this second analogy we may consider an arbitrary adelic connection $\delta = (\delta_p)$, with attached Frobenius lifts (ϕ_p), and we may define the $(1,1)$-*curvature* of δ as the matrix of "divided commutators" $(\varphi_{p\bar{p}'})$

$$\varphi_{p\bar{p}'} := \frac{1}{pp'}[\phi_p, \phi_{0p'}] : A[[T]] \to A[[T]], \quad p \neq p', \tag{47}$$

$$\varphi_{p\bar{p}} := \frac{1}{p}[\phi_p, \phi_{0p}] : A[[T]] \to A[[T]]. \tag{48}$$

• Going back to our discussion of curvature for Chern connections consider, again, $q \in GL_n(A)$ with $q^t = \pm q$.

• One can show that if all the entries of q are roots of unity or 0 then the Chern connection δ attached to q is global along 1; in particular δ has a well defined curvature and $(1,1)$-curvature.

• Let us say that a matrix $q \in GL_n(A)$ is *split* if it is one of the following:

$$\begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix}, \tag{49}$$

where 1_r is the $r \times r$ identity matrix and $n = 2r, 2r, 2r+1$ respectively. Let q be split and let $(\varphi_{pp'})$ be the curvature of the Chern connection on G attached to q. We can prove various vanishing/non-vanishing results for curvature and $(1,1)$-curvature; here is a sample:

- Assume $n \geq 4$. Then for all $p \neq p'$ we have $\varphi_{pp'} \neq 0$.
- Assume $n = 2r \geq 2$. Then for all p, p' we have $\varphi_{pp'}(T) \equiv 0 \mod (T)^3$.
- Assume $n = 2$ and $q^t = -q$. Then for all p, p' we have $\varphi_{pp'} = 0$.
- Assume $n \geq 2$. Then for all p, p' we have $\varphi_{p\overline{p}'} \neq 0$.
- Assume $n = 1$. Then for all p, p' we have $\varphi_{pp'} = \varphi_{p\overline{p}'} = 0$.

The first assertion morally says that *Spec* $\mathbb{Z}$ is "curved," while the second assertion morally says that *Spec* $\mathbb{Z}$ is only "mildly curved." Note that the above assertions say nothing about the vanishing of the curvature $\varphi_{pp'}$ in case $n = 2, 3$ and $q^t = q$; our method of proof does not seem to apply to these cases.

• Assume q split and $n \geq 4$ is even. Then, for the Chern connection attached to q, the following hold:

1) $\widehat{\mathfrak{hol}}$ is non-zero and pronilpotent.

2) $\mathfrak{hol}_{\mathbb{Q}}$ is not spanned over $\mathbb{Q}$ by the components of the curvature.

• Assertion 1 is in stark contrast with the fact that *holonomy Lie algebras* arising from Galois theory are never nilpotent unless they vanish. Assertion 2 should be viewed as a statement suggesting that the flavor of our arithmetic situation is rather different from that of classical *locally symmetric* spaces; indeed, for the latter, the Lie algebra of holonomy *is* spanned by the components of the curvature.

• Similar results proved for the curvature of Lax connections.

• The open sets G^* and G^{**} where isospectral and isocharacteristic Lax connections are defined (cf. 43 and 42) do not contain the identity of the group $G = GL_n$ hence curvature cannot be defined by analytic continuation along the identity; however these open sets contain certain torsion points of the diagonal maximal torus of G and we can use analytic continuation along such torsion points to define curvature and $(1,1)$-curvature. We can then prove:

• For the isocharacteristic Lax connection and $n = 2$:

$$\varphi_{p\bar{p}} \neq 0, \quad \text{all} \quad p.$$

On the other hand, by definition,

• For the (canonical) isospectral Lax connection

$$\varphi_{pp'} = 0, \quad \text{all} \quad p, p'.$$

• The concept of curvature discussed above was based on what we called *analytic continuation between primes*; this was the key to making Frobenius lifts corresponding to different primes act on a same ring and note that it only works for adelic connections that are global along 1 (or, as we shall see in the body of the book, along certain tori in GL_n). There is a different approach towards making Frobenius lifts comparable; this approach is based on *algebraizing Frobenius lifts via correspondences* and works for adelic connections that are not necessarily global along 1 (or along a torus). The price to pay for allowing this generality is that endomorphisms (of $A[[T]]$) are replaced by correspondences (on GL_n).

• Let $\delta = (\delta_p)$ be the Chern connection on $G = GL_n$ attached to a matrix $q \in GL_n(A)$ with $q^t = \pm q$. Then we prove that there exist maps of A-schemes $\pi_p : Y_p \to G$ and $\varphi_p : Y_p \to G$ such that π_p are affine and étale, the p-adic completions of π_p,

$$\pi_p^{\widehat{p}} : Y_p^{\widehat{p}} \to G^{\widehat{p}},$$

are isomorphisms and we have equalities of maps,

$$\varphi_p^{\widehat{p}} = \phi_p \circ \pi_p^{\widehat{p}} : Y_p^{\widehat{p}} \to G^{\widehat{p}}.$$

In other words the *correspondences*

$$\Gamma_p := (Y_p, \pi_p, \varphi_p)$$

on G are "algebraizations" of our Frobenius lifts ϕ_p. The family (Γ_p) will be referred to as a *correspondence structure* for (δ_p). This structure is not unique but does have some "uniqueness features."

• On the other hand any correspondence Γ_p acts on the field E of rational functions of $G = GL_n$ by the formula $\Gamma_p^* : E \to E$,

$$\Gamma_p^*(z) := \mathrm{Tr}_{\pi_p}(\varphi_p^*(z)), \quad z \in E, \tag{50}$$

where $\mathrm{Tr}_{\pi_p} : F_p \to E$ is the trace of the extension $\pi_p^* : E \to F_p := Y_p \otimes_G E$ and $\varphi_p^* : E \to F_p$ is induced by φ_p. So one can define the *curvature* of (Γ_p) as the matrix $(\varphi_{pp'}^*)$ where

$$\varphi_{pp'}^* := \frac{1}{pp'}[\Gamma_p^*, \Gamma_{p'}^*] : E \to E, \quad p, p' \in \mathcal{V}. \tag{51}$$

• Note that, in this way, we have defined a concept of "curvature" for Chern connections attached to arbitrary q's (that do not necessarily have entries zeroes or roots of unity). There is a $(1,1)$-version of the above as follows. Indeed the trivial adelic connection $\delta_0 = (\delta_{0p})$ has a canonical correspondence structure (Γ_{0p}) given by

$$\Gamma_{0p} = (G, \pi_{0p}, \varphi_{0p}),$$

where π_{0p} is the identity, and $\varphi_{0p}(x) = x^{(p)}$. One can then define the $(1,1)$-*curvature* of (Γ_p) as the family $(\varphi^*_{p\overline{p}'})$ where $\varphi^*_{p\overline{p}'}$ is the additive endomorphism

$$\varphi^*_{p\overline{p}'} := \frac{1}{pp'}[\Gamma^*_{0p'}, \Gamma^*_p] : E \to E \quad \text{for } p \neq p', \tag{52}$$

$$\varphi^*_{p\overline{p}} := \frac{1}{p}[\Gamma^*_{0p}, \Gamma^*_p] : E \to E. \tag{53}$$

For q split we have:

• Assume $n = 2$ and $q^t = -q$. Then for all p, p' we have $\varphi^*_{pp'} = 0$ and $\varphi^*_{p\overline{p}'} \neq 0$.
• Assume $n = 2$ and $q^t = q$. Then for all p, p' we have $\varphi^*_{p\overline{p}'} \neq 0$.

Once again our results say nothing about curvature in case $n = 2$ and $q^t = q$; our method of proof does not seem to apply to this case.

• For the Levi-Civita connection $(\delta_{1p}, ..., \delta_{np})$ attached to a symmetric q one can define a *curvature* (φ_p^{ij}), indexed by $i, j = 1, ..., n$ given by the divided commutators

$$\varphi_p^{ij} := \frac{1}{p}[\phi_{ip}, \phi_{jp}] : \mathcal{O}(G^{\widehat{p}}) \to \mathcal{O}(G^{\widehat{p}}). \tag{54}$$

This is a "vertical" curvature (indexed by the index set of the columns and rows of x) rather than a "horizontal" curvature, in the style of the previously introduced curvatures (which are indexed by primes). We can prove non-vanishing results for these curvatures. For instance assume i, j, k, l are fixed indices between 1 and n. We have:

• Assume $\delta_p q_{jk} + \delta_p q_{il} \not\equiv \delta_p q_{ik} + \delta_p q_{jl}$ mod p. Then $\varphi_p^{ij}, \varphi_p^{kl} \not\equiv 0$ mod p.
• Assume $n = 2r$ and q is split. Then $\varphi_p^{ij} \not\equiv 0$ mod p for $i \neq j$.

• We can prove a more precise result. Set

$$\Phi^{ij} := \Phi^{ij}(x) := \varphi^{ij}(x), \quad \Psi_{ij} := \Psi_{ij}(x) := x^{(p^2)t} q^{(p^2)} \Phi^{ij}(x),$$

and let R_{ijkl} be the (k, l)-entry of the matrix Ψ_{ij}, so

$$\Psi_{ij} = (R_{ijkl}). \tag{55}$$

Then the R_{ijkl} in 55 can be viewed as an arithmetic analogue of the covariant Riemann tensor in classical differential geometry.

• We can prove that (R_{ijkl}) in 55 satisfies congruences mod p:

$$R_{ijkl} \equiv -R_{jikl}, \quad R_{ijkl} \equiv -R_{ijlk}, \quad R_{ijkl} + R_{iklj} + R_{ijlk} \equiv 0, \quad R_{ijkl} \equiv R_{klij}, \tag{56}$$

which are, of course, an arithmetic analogue of symmetries of Riemann tensor.

• In addition we can prove the congruence

$$R_{ijkl} \equiv -\frac{1}{2}(\delta q_{jk} + \delta q_{il} - \delta q_{ik} - \delta q_{jl})^p \mod (p, x-1), \tag{57}$$

equivalently the congruence

$$R_{ijkl} \equiv -\frac{1}{2}(\delta(p\delta q_{jk}) + \delta(p\delta q_{il}) - \delta(p\delta q_{ik}) - \delta(p\delta q_{jl})) \mod (p, x-1). \tag{58}$$

These can be viewed as one of the following:

1) an "infinitesimal" analogue of the condition of constant sectional curvature;

2) an analogue of the formula giving curvature of normal metrics at the origin.

• For a fixed p and the Fedosov connection $(\delta_{1p}, \delta_{2p})$ relative to any antisymmetric $q \in GL_2(A)$ the formula 54 defines, again, a *curvature*; we can prove that this curvature does not vanish in general even if q is split.

Part II: Comparison with other theories

• A number of analogies between primes and geometric objects have been proposed. Here are three of them:

A) Primes are analogous to points on a Riemann surface.
B) Primes are analogous to knots in a 3-dimensional manifold.
C) Primes are analogous to directions in an infinite dimensional manifold.

• The viewpoint A is classical, going back to Dedekind, Hilbert, etc. The framework of Grothendieck, Arakelov, etc., also fits into viewpoint A. According to this viewpoint the ring of integers $\mathbb{Z}$, or more generally rings of integers in number fields, can be viewed as analogues of rings of functions on Riemann surfaces or affine algebraic curves; these are objects of complex dimension 1 (or real dimension 2). Genera of number fields are classically defined and finite, as in the case of Riemann surfaces. There is a related viewpoint according to which $\mathbb{Z}$ is the analogue of an algebraic curve of infinite genus; cf. e.g., Connes-Consani.

• The viewpoint B originates in suggestions of Mazur, Manin, Kapranov, and others. According to this viewpoint *Spec* $\mathbb{Z}$ should be viewed as an analogue of a 3-dimensional manifold, while the embeddings *Spec* $\mathbb{F}_p \to$ *Spec* $\mathbb{Z}$ should be viewed as analogues of embeddings of circles. The Legendre symbol is then an analogue of linking numbers. This analogy goes rather deep; cf. Morishita.

• Our approach adopts the viewpoint C.

• There are other approaches that adopt the viewpoint C. For instance Haran's theory (and previous work of Kurokawa and others) considers the operators

$$\frac{\partial}{\partial p} : \mathbb{Z} \to \mathbb{Z}, \quad \frac{\partial a}{\partial p} := v_p(a)\frac{a}{p},$$

where $v_p(a)$ is the p-adic valuation of a. These operators have a flavor that is rather different from that of Fermat quotients, though, and it seems unlikely that Haran's theory and ours are directly related.

• Borger's philosophy on $\mathbb{F}_1$ can also be viewed as a viewpoint consistent with C above. Indeed Borger's beautiful suggestion is to take λ-*geometry* as a possible candidate for the *geometry over the field with one element*, $\mathbb{F}_1$; this viewpoint is perpendicular to other approaches to $\mathbb{F}_1$ (such as that of Connes-Consani, Haran, etc.) but is consistent, in certain spacial cases, with our theory. Recall that λ-*geometry* is, essentially, the usual algebraic geometry of schemes X equipped with a commuting family (ϕ_p) of Frobenius lifts $\phi_p : X \to X$. So our theory fits into λ-geometry as long as:
1) the Frobenius lifts are defined on the schemes X themselves (rather than on the various p-adic completions $X^{\widehat{p}}$) and
2) the Frobenius lifts commute.
However conditions 1 and 2 are almost never satisfied in our theory: the failure of condition 2 is precisely the origin of our curvature, while finding substitutes for condition 1 requires taking various convoluted paths (such as analytic continuation between primes or algebraization by correspondences). So in practice our approach places us, most of the times, outside the paradigm of λ-geometry.

• Next we would like to point out what we think is an important difference between our viewpoint here and the viewpoint proposed by Ihara. Our approach, in its simplest form, proposes to see the operator

$$\delta = \delta_p : \mathbb{Z} \to \mathbb{Z},\ a \mapsto \delta a = \frac{a - a^p}{p},$$

where p is a fixed prime, as an analogue of a derivation with respect to p. Ihara proposed to see the map

$$d : \mathbf{Z} \to \prod_p \mathbb{Z}/p\mathbb{Z},\ a \mapsto \left(\frac{a - a^p}{p}\ mod\ p \right) \tag{59}$$

as an analogue of differentiation for integers and he proposed a series of conjectures concerning the “zeroes” of the differential of an integer. These conjectures are still completely open; they are in the spirit of the approach A listed above, in the sense that counting zeroes of 1-forms is a Riemann surface concept. But what we see as the main difference between Ihara's viewpoint and ours is that *we do not consider the reduction mod p of the Fermat quotients but the Fermat quotients themselves*. This allows the possibility of considering compositions between our δ_p's which leads to the possibility of considering arithmetic analogues of differential equations, curvature, etc.

• It is worth pointing out that what we call *curvature* in our context is entirely different from what is called *p-curvature* in the arithmetic theory of differential equations that has been developed around the Grothendieck conjecture; indeed our curvature here is about the "p-differentiation" of numbers with respect to primes p (in other words it is about d/dp) whereas the theory in Katz' papers (and related papers) is about usual differentiation d/dt with respect to a variable t of power series in t with arithmetically interesting coefficients. A similar remark can be made about the difference between our approach and that in Bost's work. In spite of this contrast there could be relations among these two types of curvatures; indeed the arithmetic direction d/dp (which is the one we are interested in our work) and the geometric direction d/dt (involved in Grothendieck's conjecture) *do* interact in interesting ways.

• Finally we would like to point out that the theory in this book is a priori unrelated to topics such as *the geometry of numbers* on the one hand and *discrete differential geometry* on the other. Indeed in both these geometries what is being studied are discrete configurations of points in the Euclidean space $\mathbb{R}^m$; in the geometry of numbers the configurations of points typically represent rings of algebraic numbers while in discrete differential geometry the configurations of points approximate smooth submanifolds of the Euclidean space. This framework is, therefore, that of the classical geometry of Euclidean space, based on $\mathbb{R}$-coordinates, and *not* that of an analogue of this geometry, based on "prime coordinates." It may very well happen, however, that (one or both of) the above topics are a natural home for some (yet to be discovered) Archimedian counterpart of our (finite) adelic theory.

Part III: Open problems

Unifying $\mathfrak{hol}_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$

• We would like to view the Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$ as an infinitesimal analogue of the absolute Galois group $\Gamma_{\mathbb{Q}} = Gal(\mathbb{Q}^a/\mathbb{Q})$.

• On the other hand if Hol is the holonomy group of a connection on principal G-bundle over a manifold M in classical differential geometry and if Hol^0 is its connected component then the quotient Hol/Hol^0 is usually referred to as the *monodromy group* of the connection; it is isomorphic to the image Γ_M of the monodromy representation

$$\pi_1(M) \to G \tag{60}$$

defined by our connection. (We have ignored here the base points.)

• The exact sequence

$$1 \to Hol^0 \to Hol \to \Gamma_M \to 1 \quad (61)$$

gives rise to a natural homomorphism

$$\Gamma_M \to \mathrm{Out}(Hol^0). \quad (62)$$

• It is classical to see the absolute Galois group $\Gamma_{\mathbb{Q}}$ as an arithmetic analogue of a fundamental group $\pi_1(M)$. Galois representations are then analogous to monodromy representations 60 and their images are analogous to the groups Γ_M. Our Lie algebras $\mathfrak{hol}_{\mathbb{Q}}$ attached to real Chern or Levi-Cività adelic connections are an arithmetic analogue of the Lie algebra of Hol^0. Also one can naturally attach Lie algebras to data from Galois theory ("Galois connections"). It is reasonable to pose the following:

Problem 1. "Unify" our holonomy on GL_n with Galois theory by constructing canonical extensions of Lie algebras

$$0 \to \mathfrak{hol}_1 \to \mathfrak{hol}_2 \to \mathfrak{hol}_3 \to 0 \tag{63}$$

where $\mathfrak{hol}_3$ is attached to a Galois connection and $\mathfrak{hol}_1$ is attached to a real Chern or Levi-Cività (adelic) connection on GL_n. Such an extension 63 should function as an arithmetic analogue of the extension 61 and the induced representation

$$\mathfrak{hol}_3 \to \mathrm{Out}(\mathfrak{hol}_1) \tag{64}$$

should be an analogue of the representation 62. This construction could involve interesting Galois representations; indeed an extension such as 63 could be, in its turn, analogous to the basic exact sequence

$$1 \to \pi_1(X^a) \to \pi_1(X) \to \Gamma_{\mathbb{Q}} \to 1 \tag{65}$$

attached to a geometrically connected scheme X over $\mathbb{Q}$ (where $X^a = X \otimes \mathbb{Q}^a$ and we suppressed, again, reference to base points).

The representation 64 could then be analogous to the representation

$$\Gamma_{\mathbb{Q}} \to \mathrm{Out}(\pi_1(X^a)) \tag{66}$$

arising from 65. The representations 66 play a prominent role in work of Grothendieck, Ihara, Deligne, and many others.

Here are some related problems:

Problem 2. Find links between our curvature of adelic connections and reciprocity (as it appears in arithmetic topology); the appearance of the Legendre symbol in our Christoffel symbols may be an indication that such links may exist.

Problem 3. Develop a Galois correspondence for Ehresmann connections; extend the Galois theory from Ehresmann connections to other types of connections, especially real Chern and special linear connections.

Problem 4. Find links between our adelic (flat) connections and Galois representations similar to the classical link between vector bundles with flat connections and monodromy.

Problem 5. Find an arithmetic analogue of Atiyah's theory according to which:
1) the algebraic connections on an algebraic vector bundle over a variety are in bijection with the splittings of the Atiyah extension;
2) the obstruction to the splitting of the Atiyah extension generates the characteristic ring of the vector bundle.

Problem 6. The curvature of adelic Chern connections led, in our theory, to objects that we called *Chern* $(1,1)$-*forms*; it is tempting to use these forms, and higher versions of them, to define cohomology classes in the style of Chern-Weil theory; the target cohomology groups could be Galois (étale) cohomology groups of appropriate arithmetic objects.

Unifying "$\partial/\partial p$" and "$\partial/\partial \zeta_{p^\infty}$"

The theory of p-adic periods of Fontaine and Colmez involves, in particular, a "totally ramified arithmetic calculus" that is perpendicular to our (unramified) arithmetic calculus. In its simplest form this totally ramified calculus morally comes from the fact that the modules of Kähler differentials $\Omega_{\mathbb{Z}[\zeta_{p^n}]}$ are torsion but non-trivial: $d\zeta_{p^n} \neq 0$. It is conceivable that this situation could lead to a "derivation like" operator, "$\partial/\partial \zeta_{p^\infty}$" on an appropriate space on which our p-derivation δ_p also acts. Then one is led to:

Problem 7. Unify our unramified arithmetic calculus (based on p-derivations δ_p, thought of as "$\partial/\partial p$") with the totally ramified arithmetic calculus of Fontaine-Colmez (possibly involving the conjectural "derivation like" operators "$\partial/\partial_{p^\infty}$"). One would presumably get, for each p, a partial differential situation with two "directions" and, in particular, one could attempt to study the resulting "curvature." A way to proceed could be to start from an earlier paper where the base ring was the ring of power series $R[[q]]$ in the variable q, equipped with the "standard" p-derivation δ_p (extending that of $R = W(\mathbb{F}_p^a)$ and killing q) and with the usual derivation $\delta_q = qd/dq$. Then one could try to specialize the theory in that paper via $q \mapsto \zeta_{p^n}$ with $n \to \infty$. Such a specialization does not a priori make sense, of course, and no recipe to give it a meaning seems available at this point. However an indication that this procedure might be meaningful can be found in the formulas giving the "explicit reciprocity laws" of local class field theory due to Shafarevich, Brückner, Vostokov, etc.

Unifying Sh and GL_n

Problem 8. Relate the arithmetic differential calculus on Shimura varieties Sh (B, AMS 2005) with the arithmetic differential calculus on the classical groups (appearing in our present book). In particular examine the case of

$$Sh = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R}).$$

An indication that such a relation may exist is the δ-algebraic theory of δ-Hodge structures in (AB, AJM 1995) giving δ-algebraic correspondences between classical groups and moduli spaces of abelian varieties; one would have to develop an arithmetic analogue of that paper.

Here are some related questions:

Problem 9. Find a link between the concept of curvature in the present book and the concept of curvature based on the arithmetic Laplacian (AB+S.Simanca, Advances, 2008). More generally find a common ground between our theory here and that paper.

Problem 10. The theory in this book is mainly about adelic connections on GL_n possibly compatible with involutions of GL_n. It is natural to explore what happens if one replaces GL_n by a general reductive group.

Problem 11. Explore the arithmetic differential geometry of homogeneous spaces such as "spheres," $S^n := SO_{n+1}/SO_n$.

Problem 12. Develop an arithmetic analogue of the Cassidy-Kolchin theory of differential algebraic groups. Compute the rings of invariant δ-functions for actions of such groups on the varieties naturally appearing in Riemannian geometry. For instance compute the δ-functions on $SO_n\backslash GL_n/\Gamma$ where Γ is a δ-subgroup of GL_n. The δ-invariants of the curvature of the Levi-Cività connection may be playing a role in this problem.

Problem 13. Find interactions between the classical theory of quadratic forms over rings of integers in number fields and our theory here (around symmetric matrices that play the role of metrics). Both theories are about the geometry of GL_n/SO_n.

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