Algebraic curves (graduate course)

Plan.

0. Introduction and program.
1. Rings and modules. Localisation.
2. Integral dependence and dimension
3. Noetherian rings. (Dedekind rings) Discrete valuation rings.
4. Algebraic sets in $\mathbb{A}^n$ and $\mathbb{P}^n$
5. Algebraic sets in $\mathbb{A}^n \times \mathbb{P}^m$ and $\mathbb{P}^n \times \mathbb{P}^m$
6. Algebraic varieties
7. Complements about morphisms
8. Non-singular projective models of curves
9. Divisors on curves. Bézout theorem.
10. Curves in projective plane
11. Differentials. Genus and canonical class
12. Linear systems and Riemann-Roch theorem.
13. Hurwitz theorem. Elliptic curves.
14. Hyperelliptic and canonical curves.
15. Proof of Riemann-Roch I: sheaves, cohomology
16. Proof of Riemann-Roch II: residues
17. Proof of Riemann-Roch III: adèles and duality.

?

⋮

(1–3) 3 lectures on Comm. Algebra cf. Atiyah-Macdonald

(4–7) 4 lectures on Alg. varieties (generalities) cf. Šaf. & Hartshorne ch 1.

(8–14) 7 lectures on Alg. curves (with RR assumed but not proved) cf. Šaf & Hartsh (ch. on curves)

(15–17) 3 lectures on Pf of RR for curves cf Serre.

x lectures on Jacobians

⋮

Have to change time. Only pm

Aim: introd to one of subj in heart of alg. geom: A.C. ← prereq. have to know what's a ring, a field, an alg. cl. [illegible]

Lecture 0. Introduction and Program

references ← Har, Saf, Serre. Gr alg & corps cl.

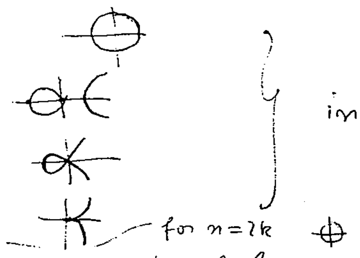
Let k be an alg. closed field ($\mathbb{C}$, $\overline{\mathbb{F}_p}$, ...) [Nevertheless in drawing pictures we abusively take $k=\mathbb{R}$ which is not alg. cl. Pictures will nevertheless help intuition].

Affine plane $A^2 = k^2 = k \times k$, coord sys

Let $f \in k[x,y]$ a polyn.

Affine plane curve $X = \{(a,b) \in k^2 \mid f(a,b)=0\}$ $(= Z(f))$

f is called irred if $f=gh \Rightarrow g \in k$ or $h \in k$; X is called irred if f is so



Ex. I $y^2 + x^2 - 1 = 0$

II $y^2 - x + x^3 = 0$

III $y^2 - x^2 - x^3 = 0$

IV $y^2 - x^3 = 0$

V $x^n + y^n = 1$, $n \geq 3$ — for $n = 2k$

in $\mathbb{R}^2$ (not $\mathbb{C}^2$!)

(Def) X is called unirational if $\exists$ rational functions $\varphi, \psi \in k(t)$, $\varphi = \frac{F}{G}$, $\psi = \frac{H}{K}$, $F, G, H, K \in k[t]$ polyn. (not both φ & ψ constant) s.t. $f(\varphi(t), \psi(t)) = 0$ for all $t \in k \setminus \underbrace{\{\text{roots of } G\} \cup \{\text{roots of } K\}}_{S}$. In other words if $\varphi \times \psi : k \setminus S \to k^2$ takes its values in X. Intuitively $\exists$ "rational parametrisation" (x-line with pts deleted). X is called rational if in addition we impose $\varphi \times \psi : k \setminus S \to k^2$ be injective.

(Rem) A key result of the theory is (unirat $\Rightarrow$ rat) (different par makes it rat.!)

One of the main questions: decide when is X rational.

(Ex) I is rat $\varphi(t) = \frac{2t}{t^2+1}$, $\psi(t) = \frac{t^2-1}{t^2+1}$ (it has to do with $\sin \alpha = \frac{2 \tan\frac{\alpha}{2}}{\ldots}$)

II is not (it is not at all easy to prove this! we shall prove it later)

III is rat $\varphi(t) = t^2 - 1$, $\psi(t) = t(t^2-1)$ [ $y = tx$]

IV is rat $\varphi(t) = t^2$, $\psi(t) = t^3$

V is not (hard!)

(Why) we are concerned with rationality? At least 2 reasons:

1) an arithmetic reason: Ass f has coeff in $\mathbb{Q}$ ~~[illegible]~~ Then one can ask: is $X \cap \mathbb{Q}^2$ infinite? (Does the eqn $f(x,y)=0$ have ∞ sol $(x,y) \in \mathbb{Q}^2$?). Clearly if X is rat$^{(*)}$ $\Rightarrow$ $X \cap \mathbb{Q}^2$ is infinite ($\forall t \in \mathbb{Q}$, $(\varphi(t), \psi(t)) \in X \cap \mathbb{Q}^2$)!

On the other hand we have Mordell's conj (th of Falt). If X is not rat. and not elliptic (this concept will be defined in the course) $\Rightarrow$ $X \cap \mathbb{Q}^2$ is finite! ← FIELDS MEDAL ← Not to be proved in the course.

(*) in add φ & ψ have coeff in $\mathbb{Q}$!

2) an analytic reason. Assume we want to compute $\int g(x, u(x))dx$ where $u(x)$ is an alg fun i.e. $\exists$ polyn $f \in k[x,y]$ s.t. $f(x,u(x))=0$. Assume $X = \{f=0\}$ is rational with param. (φ, ψ). Then $\int g(x, u(x))dx \overset{t}{=}$

$= \int \underbrace{g(\varphi(t), \psi(t))\,\psi'(t)}_{\text{rational fun of } t}dt$ ← can be computed as known from calculus as compos of rat, ln, arctg

E.g. For $\int g(x, \sqrt{ax^2+bx+c})dx$ let $f(x,y) = y^2 - (ax^2+bx+c) \Rightarrow$ Euler substit.

Remark $\int \frac{dx}{\sqrt{P(x)}}$ (**) (P of deg 3 with distinct roots) "does not reduce" to "known funs" (rat, ln, arctg). Nevertheless starting from this integr one is led to a "new trigonometry" (new = 100 years old).

Generalis of the rationality problem. Two curves $Z(f)$ & $Z(g)$ are called bir equiv if ⋯ Basic problem: find theoretical & practical ways of deciding when two curves are bir equiv (a curve is rat iff it is bir eq to a line)

Program for curves 1) One has to pass from the aff plane curve $X = \{f(x,y)=0\} \subset \mathbb{A}^2$ to the plane proj curve $\overline{X} = \{F(x,y,z)=0\} \subset \mathbb{P}^2$ (***), F = homog. of f [e.g. if $f = y^2 = x - x^3 \Rightarrow F = y^2z = xz^2 - x^3$]; $\overline{X} = X \cup \{$finitely many pts at $\infty\}$.

2) One has to pass from the plane proj curve $\overline{X}$ to the non-sing proj curve $\tilde{X}$ in $\mathbb{P}^n$ (called its normalization) (n may be big) by an algebraic procedure (4) ~~called ...~~.

3) One has to study "differentials" on $\tilde{X}$ [$\Leftrightarrow$ integrals $\int g(x,u(x))dx$]

This leads to attaching to each non-sing proj curve a number $g \geq 0$ called genus. $\tilde{X}_1$ and $\tilde{X}_2$ have diff genera $\Rightarrow X_1, X_2$ are not bir. isomorphic

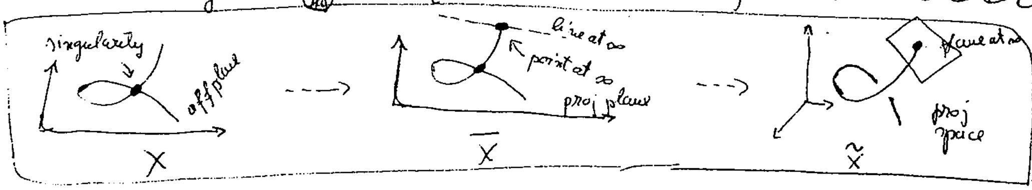
finer study: 4) One has to study "divisors" on $\tilde{X}$ [a div $D = \sum n_i P_i$, $n_i \in \mathbb{Z}$, $P_i \in X$ & "lin equivalence"]. Here th of RR.

5) $\Rightarrow$ $g=0$ iff X is rational (bir. to line)

$g=1$ iff X is "elliptic" (bir eq to $y^2 = P(x)$, $\deg P = 3$ with no mult roots) $\exists$ group law $\tilde{X}\times\tilde{X}\to\tilde{X}$ defined by "rational formulas"

$g \geq 2$ $\Rightarrow$ X is of 2 possible types: hyperell (described) and "canonical" $\tilde{X} \subset \mathbb{P}^{g-1}$. Two such curves are bir equiv $\Leftrightarrow$ the two $\tilde{X}$ are proj eq.

$y^2 = x(x-1)(x-\lambda)$. Put $j(\lambda) = \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$. X_λ bir to X_μ iff $j(\lambda)=j(\mu)$



(4) This is a concept of comm algebra and will force us to develop some comm alg especially DVR

(***) $\mathbb{P}^2$ will be defined as $k^3 \setminus \{0\}/\sim$ where $\sim$ is ⋯, $\mathbb{P}^2 = \mathbb{A}^2 \cup$ line at ∞; $\mathbb{P}^n = k^n \cup$ hyp at ∞

(**) corr to the non-rat curve $y^2 - P(x) = 0$.

(t) $f(\varphi(t), \psi(t)) = f(\varphi(t), u(\varphi(t))) = 0$ so $u(\varphi(t))$ and $\psi(t)$ are "equal up to permut of roots". We assume here they are $=$.

Lecture 1 ~~Review of~~ Rings, Modules, Localisation.

$(A,+)$ abel. gr. $(A,\cdot)$ ass. with 1-elt, distrib. conditions

(add examples in $k[x,y,z]$ (and its quotients) after each Def) $(A,+,\cdot)$

Def "ring" will mean comm. ring with 1-element (not necess. $1\neq 0$!)

"ring homo" $f: A \to B$ will mean "f comm with $+,\cdot$" and $f(1)=1$.
(by an A-algebra we mean simply a ring homo $A\to B$).
"subring" means ...
def of product of rings!

Def $\mathfrak{a} \subset A$ ideal if $<$ subgr; $xy\in\mathfrak{a}$, $\forall x\in A, y\in\mathfrak{a}$; ideal gen by $f_1,\dots,f_n := \{\sum g_i f_i\}_{g_i\in A}$ $(f_1,\dots,f_n)\subset A$; (f) called princ. ideal.

ex $(x), (x^2-y^2)\subset k[x,y,z]$.

The quot group $A/\mathfrak{a}$ becomes a ring; its elts $x+\mathfrak{a}$ or $\bar{x}$, ex: $k[x,y,z]/(x^2-y^2)$

If $f: A\to B$ homo $\Rightarrow f^{-1}(\text{ideal}) = \text{ideal}$.

Prop $\exists$ one to one corr bet $\{$ideals of $A/\mathfrak{a}\}$ and $\{$id of A cont $\mathfrak{a}\}$.

Pf: $\pi: A\to A/\mathfrak{a}$; $\mathfrak{b}\mapsto \pi(\mathfrak{b}) = \mathfrak{b}/\mathfrak{a}$, $\mathfrak{c}\mapsto \pi^{-1}(\mathfrak{c})$.

Def $x\in A$ is called a zero divisor / unit / nilp. elt if ---
ex $k[x,y]/(xy)$, $k[x]/(x^n)\ni \bar{x}$, $k[x,y,z]/(p,q-1)$.
$A^* =$ set of units of A (a mult. group).
A is called an integral domain if $\nexists$ zero divisors (and $A\neq 0$) ex $k[x,y]$.
A is called reduced if ...
A called a field if ...

Prop $A\neq 0$ is a field $\Leftrightarrow$ the only ideals are (0) & (1). Ex $\mathbb{Q},\mathbb{R},\mathbb{C},\mathbb{F}_p$.

Def $\mathfrak{p}\subset A$ is called prime if $\mathfrak{p}\neq A$ & $(xy\in\mathfrak{p}\Rightarrow x\in\mathfrak{p}$ or $y\in\mathfrak{p})$; Spec A

$\mathfrak{m}\subset A$ — " — max if $\mathfrak{m}\neq A$ & $(\mathfrak{m}\subset\mathfrak{n}\underset{\neq}{\subset} A\Rightarrow \mathfrak{m}=\mathfrak{n})$; Max A

Def A is called local if it has only one max ideal.

Prop $\mathfrak{p}$ prime $\Leftrightarrow A/\mathfrak{p}$ integr. domain   ex (x,y) prime in $k[x,y,z]$ bec $\simeq k[z]$

$\mathfrak{m}$ max $\Leftrightarrow A/\mathfrak{m}$ field. (x,y,z) max in — " — $\simeq k$.

More gen. $(x-a, y-b)$ instead of (x,y).

Prop $f: A\to B$ homo $\Rightarrow f^{-1}(\text{prime}) = \text{prime}$ [but $f^{-1}(\text{max})$ may not be max. $(\mathbb{Z}\to\mathbb{Q}\ni(0))$]

Prop Any ideal $\mathfrak{a}\subset A$ is contained in a max ideal.

Pf: Apply Zorn lemma (any non-empty partially ordered set - i.e. $(\Sigma,\leq)$ with $x\leq x$, $(x\leq y$ & $y\leq z\Rightarrow x\leq z)$, $(x\leq y$ & $y\leq x\Rightarrow x=y)$ - s.t. any totally ordered subset in Σ has an upper bound, has a maximal elt) to the set $\{\mathfrak{b}\subset A \mid \mathfrak{b}$ ideal $\supset\mathfrak{a}, \mathfrak{b}\neq A\}$.

Ex If $A=\mathbb{Z}$, (n) prime $\Leftrightarrow n$ is prime or $n=0$ [all $(n)\neq(0)$ primes are maximal!]

Def A is called factorial if it is an integr dom & any elt ($\neq 0, \notin A^*$) is a $\prod$ of prime elts (an elt f is prime iff (f) is a prime ideal i.e. if $f\mid gh\Rightarrow f\mid g$ or $f\mid h$).
An elt is called irred if ... In a fact ring irred $\Leftrightarrow$ prime.

NP **Prop** k field $\Rightarrow k[x_1,\dots,x_n]$ factorial (Gauss theorem See Zar & Samuel)

Cor If $f\in k[x_1,\dots,x_n]$ irred $\Rightarrow (f)$ prime.
Eisenstein criterion of irreducibility e.g. $x^n+y^n+z^n$ irr, y^2-x^3+x irr) Pf... (in $A[x]$, A fact)

Def $nil(A) = \{x\in A \mid x$ is nilp $\}$

Prop $nil(A)$ is an ideal and $A/nil(A)$ is reduced

Pf. If $x^n = y^m = 0$, by the binomial th $(x+y)^{n+m-1} = 0$

Prop $nil(A) = \bigcap$ all primes of A

Pf For "$\supset$" let x ~~all primes~~ be not nilp and apply Zorn to the set Σ of ideals not containing any x^n for $n\geq 0$. Let $\mathfrak{p}$ be max elt of Σ. One checks that $\mathfrak{p}$ is prime: [If $y,z\notin\mathfrak{p}$, $yz\in\mathfrak{p}\Rightarrow x^n\in\mathfrak{p}+(y)$, $x^m\in\mathfrak{p}+(z)$ $\Rightarrow x^{n+m}\in\mathfrak{p}+(zy)=\mathfrak{p}$ ↯]

notations: rings A,B,C, elts x,y,z.
ideals $\mathfrak{a},\mathfrak{b},\mathfrak{c}$, $\mathfrak{m},\mathfrak{n}$ or $\underline{a},\underline{b},\underline{c},\underline{m},\underline{n}$
modules M,N,P.

Def For $\mathfrak{a} \subset A$ ideal define $\sqrt{\mathfrak{a}} = \{x \in A \mid \exists n \text{ s.t. } x^n \in \mathfrak{a}\}$ (so $nil(A) = \sqrt{(0)}$).

Note that $\sqrt{\mathfrak{a}} = \varphi^{-1}(nil(A/\mathfrak{a}))$, $\varphi: A \to A/nil(A)$.

Cor $\sqrt{\mathfrak{a}} = \bigcap$ all primes containing $\mathfrak{a}$ and $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.

Pf. $\varphi^{-1}(\bigcap \text{ primes of } A/\mathfrak{a}) = \bigcap \varphi^{-1}(\text{primes of } A/\mathfrak{a})$ $(\varphi^{-1}(\bigcap) = \bigcap \varphi^{-1})$.

Def $\mathfrak{a}, \mathfrak{b} \subset A$ ideals. One puts

$\mathfrak{a} + \mathfrak{b} = \{x + y \mid x \in \mathfrak{a}, y \in \mathfrak{b}\}$

$\mathfrak{a}\mathfrak{b}$ = ideal generated by $\{xy \mid x \in \mathfrak{a}, y \in \mathfrak{b}\}$

$\mathfrak{a} \cap \mathfrak{b}$ = usual

remark: $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$ $(x^n \in \mathfrak{a} \cap \mathfrak{b} \Rightarrow x^{2n} \in \mathfrak{a}\mathfrak{b})$

Def $f: A \to B$, $\mathfrak{a} \subset A$, $\mathfrak{b} \subset B$. We put $\mathfrak{a}^e \overset{not}{=} \mathfrak{a}B :=$ ideal gen by $f(\mathfrak{a})$ in B

$\mathfrak{b}^c = f^{-1}(\mathfrak{b})$.

Def An A-module is an abel. gr M plus a mapping $A \times M \to M$, $(\lambda, y) \mapsto \lambda y$ s.t. $\lambda(x+y) = \lambda x + \lambda y$, $(\lambda + \mu)x = \lambda x + \mu x$, $(\lambda\mu)x = \lambda(\mu x)$, $1x = x$.

A submodule is ... . ~~If~~ If $S \subset M$ subset one defines the submod gen by S. M is called fin gen — ?

Exam. $\mathfrak{a} \subset A$ ideal $\Rightarrow$ $\mathfrak{a}$ & $A/\mathfrak{a}$ are modules.

$\mathfrak{a}$ ideal, M module $\Rightarrow$ one defines the submodule $\mathfrak{a}M$ of M

M, N modules $\Rightarrow$ $M \oplus N$ can be given a str of module; In partic $A^{\oplus n}$ is.

N submod of M $\Rightarrow$ M/N — " — — " —

formula $S(M/N) = (SM + N)/N$

Def An A-module homo $f: M \to N$ means a homo of gr s.t $f(\lambda x) = \lambda f(x)$ $\forall \lambda \in A, x \in M$

Def. $\mathrm{Ker} f = \cdots$, $\mathrm{Im} f = \cdots$, $\mathrm{Coker} f = N/\mathrm{Im} f$ — They are submodules.

Def $M \xrightarrow{u} N \xrightarrow{v} P$ is called an ex. seq if $\mathrm{Im}\, u = \mathrm{Ker}\, v$

Def A subset $S \subset A$ is called multiplicatively closed if $1 \in S$ and $x, y \in S$ $\Rightarrow xy \in S$

Def Define the set $(A \times S)/\sim$ where $(a_1, s_1) \sim (a_2, s_2)$ iff $\exists s_3$ s.t $s_3(a_1 s_2 - a_2 s_1) = 0$

Let a/s be the eq class of (a, s). Call $S^{-1}A$ the set $(A \times S)/\sim$

Define a ring str on $S^{-1}A$ by ....: (& check it is well defined!)

[Exactly the same construction for M instead of A. We obtain $S^{-1}M$.]

We have a map $A \to S^{-1}A$, $M \to S^{-1}M$ $(x \mapsto x/1)$. $S^{-1}M$ is an $S^{-1}A$-module.

Prop (Univ prop). Let $u: A \to B$ ring homo s.t $u(S) \subset B^*$. Then $\exists$ unique morph $\sigma: S^{-1}A \to B$ s.t. the diagr ... commutes [Pf: ...]

Ex If $S = \{1, f, f^2, \ldots\}$ write $S^{-1}A = A_f$. If $S = A \setminus \mathfrak{p}$ ($\mathfrak{p}$ prime) write $S^{-1}A = A_\mathfrak{p}$.

A integral & $S = A \setminus \{0\} \Rightarrow S^{-1}A = Q(A)$ corp (de fractii). Prove formula $A_f \simeq A[X]/(fX-1)$.

Prop $M \to N \to P$ exact $\Rightarrow$ $S^{-1}M \to S^{-1}N \to S^{-1}P$ exact [Pf: ...]

Prop $M = 0$ iff $M_\mathfrak{p} = 0$ ($\forall$ prime $\mathfrak{p}$) iff $M_\mathfrak{m}$ ($\forall$ max $\mathfrak{m}$) [Pf. If $x \in M, x \neq 0$. Take $\mathfrak{m} \supset \mathrm{Ann}(x)$]

Prop $\varphi: M \to N$ is inj iff $\varphi_\mathfrak{p}: M_\mathfrak{p} \to N_\mathfrak{p}$ inj $\forall \mathfrak{p}$ iff $\varphi_\mathfrak{m}: M_\mathfrak{m} \to N_\mathfrak{m}$ inj $\forall \mathfrak{m}$ [Pf: ...]

Prop $\exists$ bij $\{$primes of A not meeting $S\}$ $\leftrightarrow$ $\{$primes of $S^{-1}A\}$ [Pf: $\mathfrak{a} \mapsto S^{-1}\mathfrak{a} = \mathfrak{a}^e$, $\mathfrak{b}^c \to \mathfrak{b}$]

Lecture 2. Integral dependence and dimension

(also say B finite /A or f.g. A-mod)

Def $A \subset B$ finite ext of rings if B is f.g. as A-mod. **Rem** $A\subset B$, $B \subset C$ fin $\Rightarrow A \subset C$ finite.

$A\subset B$ called f.g. if ... write $B = A[x_1,\dots,x_n]$, $x_i \in B$.

Def $A \subset B$ rings. An elt $x \in B$ is called integral /A if $\exists a_i \in A$ s.t. $x^n + a_1 x^{n-1} + \dots + a_n = 0$

Ex $t \in k[t]$ integral over $k[t^2,t^3]$, $\bar{y}$ integral over $k[\bar{x}]$, $\bar{y} \in k[x,y]/(y^n + a_{n-1}(x)y^{n-1} + \dots + a_0(x))$

Key lemma M a fin gen A-mod, $\mathfrak{a} \subset A$ an ideal, $\varphi: M \to M$ homo with $\varphi(M) \subset \mathfrak{a}M$. Then $\exists a_i \in \mathfrak{a}$ s.t. $\varphi^n + a_1\varphi^{n-1} + \dots + a_n I = 0$ in $\mathrm{End}_A(M)$.

Pf. Let $x_1,\dots,x_n$ gen of M, $\varphi(x_i) = \sum a_{ij} x_j$, $a_{ij} \in \mathfrak{a}$ so $\sum_j (\delta_{ij}\varphi - a_{ij})x_j = 0$. ~~Multiplying~~ Let $u_{ki} \in \mathrm{End}(M)$ s.t. (u_{ki}) is the adj of $(\delta_{ij}\varphi - a_{ij})$ $[u_{ki}$ is a polin φ with coeff in $A]$. We get $\sum_{ij} u_{ki} v_{ij} x_j = 0$. But $\sum_i u_{ki}v_{ij} = \delta_{kj} d$ where $d = \det(v_{ij})$. So $d x_j = 0$ or $d(M) = 0$. Now we expand d.

Def A local ring is ....

Cor (Nakayama's lemma). If A is $\mathfrak{m}$-local and M is fin gen s.t. $\mathfrak{m}M = M$. Then $M = 0$

Pf: Apply Key l. to $\varphi = 1_M$ and note that $1 + a_1 + \dots + a_n \in A^*$.

Nak' **Cor** A $\mathfrak{m}$-loc, M f.g./A, $x_1,\dots,x_n \in M$. Ass $\bar{x}_1,\dots,\bar{x}_n$ generate $M/\mathfrak{m}M$. $\Rightarrow x_1,\dots,x_n$ gen M [Pf: Apply Nak to $M/\sum Ax_i$]

Prop $A \subset B \ni x$. The foll are equiv:

1) x integral /A
2) $A[x]$ is a f.g. A-module (**)
3) $A[x] \subset C$ = subring of B, f.g. as an A-module.
4) $\exists$ faithful (***) $A[x]$-module M which is fin gen as an A-module.

Pf. 1)$\Rightarrow$2)$\Rightarrow$3) $\Rightarrow$4). To proove 4)$\Rightarrow$1) take φ = mult by x in M. [Pf. ind via 1)$\Rightarrow$2)]

Cor $A \subset B$; $x_i \in B$ integral /A $(1\le i \le n)$. Then $A[x_1,\dots,x_m]$ is a f.g. A-module

Cor $A\subset B$ is finite iff fin type & integral.

Cor $A \subset B$. The set C of integral elts of B over A is a subring of B [Pf. ...]

Cor $A \subset B$ integral, $B \subset C$ integral $\Rightarrow A \subset C$ integral.

Def A domain A is called integrally closed if $(\forall x \in Q(A)$, x integral /A $\Rightarrow x \in A)$.

(Ex $k[t^2,t^3]$ not int closed)

Cor A a domain. Then the ring $\tilde{A}$ of elts of $Q(A)$ which are integral /A is integrally closed (and called integr. closure of A).

Def C in ... is called the integr closure of A in B (By ... any $x \in B$ integral /C belongs to C)

Prop A fact $\Rightarrow A$ int. closed Pf .... **Cor** $k[x_1,\dots,x_n]$ int cl **Ex** $k[t]$ = int clos of $k[t^2,t^3]$.

Prop Let $A \subset B$ integral. Then

1) If $\mathfrak{b} \subset B$ ideal $\Rightarrow A/\mathfrak{b}\cap A \hookrightarrow B/\mathfrak{b}$ integral
2) If $S \subset A$ mult. closed $\Rightarrow S^{-1}A \hookrightarrow S^{-1}B$ integral

later see $| \to$ ... $k[x,y]/(y^2-x^3)$

Pf: 1) clear; 2) if $x \in B$, $x^n + a_1x^{n-1} + \dots + a_n = 0$. For $s \in S \Rightarrow (x/s)^n + (a_1/s)(x/s)^{n-1} + \dots + a_n/s^n = 0$.

Prop Let $A \subset B$ integral, both domains. Then A field $\Leftrightarrow B$ field.

Pf. "$\Rightarrow$" $x \in B \Rightarrow x^n + a_1x^{n-1} + \dots + a_n = 0 \Rightarrow x(x^{n-1} + \dots + a_{n-1}) = -a_n$; "$\Leftarrow$" Let $x \in A \Rightarrow x^{-1} \in B$ integr/A
$x^{-n} + a_1 x^{-n+1} + \dots + a_n = 0 \Rightarrow x^{-1} + \underbrace{a_1 + a_2x^2 + \dots + a_nx^{n-1}}_{} = 0 \Rightarrow x^{-1} \in A$.

Def Residue field of A at a prime $A_p/pA_p = Q(A/p)$; ex $k[x,y]$ at (x) or at (y^2-x^3). $\mathbb{Z}$ at (0), (p)

Cor Let $A \subset B$ integral. Then

1) If $q_1 \subseteq q_2 \subset B$ primes s.t. $q_1 \cap A = q_2 \cap A \Rightarrow q_1 = q_2$
2) If $q \subset B$ prime then q max $\Leftrightarrow q \cap A$ max.
3) $\forall\, p \subset A$ prime $\exists\, q \subset B$ prime s.t. $q \cap A = p$.

~~A $\subset$ B is integral if $\forall x \in B$ is integral /A ... A $\subset$ B is called finite if B is a f.g. A-mod / A $\to$ B is called of fin type if B = A[$x_1,\dots,x_n$]~~

(***) a B-module M is faithfull if $bM = 0 \Rightarrow b = 0$. In other words $B \to \mathrm{End}_B(M)$ is injective.

(**) $A[x] = \{\sum a_i x^i \mid a_i \in A\}$. More generally $A[x_1,\dots,x_n] = \{\sum a_{i_1\dots i_n} x_1^{i_1}\cdots x_n^{i_n} \mid a_{i_1\dots i_n} \in A\}$.

(*) $\mathrm{End}(M)$ = homomorph $M \to M$; it is a non-comm ring. But the polyn in φ with coeff in A form a comm ring R so theory of determ (assumed known) extends to comm rings

with $A \to B$ called integral if $\forall x \in B$ int/$\varphi(A)$

Def: quot field of an int. dom. $Q(A)$, e.g. $\mathbb{Q}$, $k(t)$

Pf. For 1) if $p = q_i \cap A \Rightarrow (A/p)_p \to (B/q_1)_p$ integral ....

For 2) $A/q\cap A \to B/q$ integral .....

For 3) . $A_p \to B_p \ni 1 \neq 0$ & take $m \subset B_p$ max. Then $m \cap A_p = pA_p$ hence $m \cap A = p$.

(Prop) $A \subset B$ rings, C = integral clos of A in B & $S \subset A$ mult syst. Then $S^{-1}C$ = int. clos of $S^{-1}A$ in $S^{-1}B$.

Pf. Ass b/s int / $S^{-1}A$ so $(b/s)^n + (a_1/s_1)(b/s)^{n-1} + \dots + a_n/s_n = 0$. Mult by $(ts)^n$ where $t = s_1 \cdots s_n \Rightarrow$ bt integr /$A \Rightarrow bt \in C \Rightarrow b \in S^{-1}C$.

(Lem) $f: M \to N$ inj, surj, bij iff f_p is so $\forall p$ iff f_m is so $\forall m$. (Ker & coker commute with fract.)

(δ) (Prop) Let A be an integral domain. Then A is integrally closed iff A_p is so for $\forall$ prime p, iff A_m is so for any max m.

~~Pf. A int cl ⇒ A_p, A_m are so by fract. prop.~~ Pf. Let C = int clos of A & $f: A \to C$. A int cl $\Leftrightarrow f: A \to C$ bij $\Leftrightarrow f_m : A_m \to C_m$ bij $\forall m$ (but C_m = int cl of A_m so q-ed).

(Def) Let A be any ring. By $\dim A$ we underst the max n s.t. $\exists$ primes $p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_n$. (chain of length n)

(Ex) $\dim(\text{field}) = 0$; $\dim \mathbb{Z} = 1$, $\dim k[x,y,z] \geq 3$ (we will see $=3$) for we have $0 \subset (x) \subset (x,y) \subset (x,y,z)$

(α) (Prop) $A \subset B$ integral $\Rightarrow \dim A = \dim B$.

Pf. Any chain in B $q_0 \subsetneq \dots \subsetneq q_m \Rightarrow q_0 \cap A \subsetneq \dots \subsetneq q_m \cap A$ so $\dim A \geq \dim B$. ($m = \dim B$)

Now let $p_0 \subsetneq \dots \subsetneq p_n \subset A$ ($n = \dim A$), we lift it inductively. Assume $\dots \subset p_i \subset p_{i+1}$, $\dots \subset q_i$.

Since $A/p_i \to B/q_i$ integr $\exists$ prime in B/q_i over $p_{i+1}/p_i \Rightarrow$ q.e.d.

(β) (Th) (Noether normalis) k an infinite field, A a f.g. k-alg. $\exists\, y_1, \dots, y_d \in A$ alg ind/k s.t. A integral / $k[y_1, \dots, y_d]$ (holds also for k finite but diff arg!)

Pf. Induction on min number of gen. Ass O.K for $\leq n-1$ & $A = k[x_1, \dots, x_n]$. If $x_1, \dots, x_n$ alg ind/k, O.K. If not (after renumbering x_i) x_n alg / $k[x_1, \dots, x_{n-1}]$, $f(x_1, \dots, x_{n-1}; x_n) = 0$

Let F be the form of highest degree in f and put $x_i = x_i' + \lambda_i x_n$ ($i < n$) where $F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$

$\Rightarrow x_n$ integral / $k[x_1', \dots, x_{n-1}']$ & apply induction. (L) $F \neq 0$ form, k infinite $\Rightarrow F$ does not van on

(δ) (Prop) If in the stat above $A = k[x_1, \dots, x_n]/(f)$, $f \neq 0$ then we can take $d = n-1$.

Pf. Let F be the form of highest deg in f, put $x_i = x_i' + \lambda_i x_n$ ($i < n$) & $x_n = x_n'$ as in ↑

This is an aut of $k[x_1, \dots, x_n]$. We may ass it is the ident. So $f = x_n^e + g_1(x_1, \dots, x_{n-1}) x_n^{e-1} + \dots$

So $\bar{x}_n \in A$ integral / $k[\bar{x}_1, \dots, \bar{x}_{n-1}]$. Now $\bar{x}_1, \dots, \bar{x}_{n-1}$ alg ind /k [if $h(\bar{x}_1, \dots \bar{x}_{n-1}) = 0 \Rightarrow h = f \cdot g$ in $k[x_1, \dots, x_n]$ & for $\deg_{x_n} h = 0$, $\deg_{x_n}(fg) \geq e$!].

(γ) (Prop) $\dim k[x_1, \dots, x_n] = n$

Pf. Ind on n. Let $d = \dim k[x_1, \dots, x_n]$, $0 = p_0 \subset p_1 \subset \dots \subset p_d$, $f \in p_1$, $f \neq 0$ $\Rightarrow f = \prod$ primes

replace f by a prime factor so ass f prime ; $(f) \subset p_1 \Rightarrow p_1 = (f)$. We have

$\dim k[x_1, \dots, x_n]/p_1 = n-1$ so the chain $\frac{p_1}{p_1} \subset \dots \subset \frac{p_d}{p_1}$ has length $\leq n-1$

(↑ ind hyp + prop (δ))

so $d - 1 \leq n-1$ so $d \leq n$. On the other hand clearly $d \geq n$ $[0 \subset (x_1) \subset (x_1, x_2) \subset \dots \subset (x_1, \dots, x_n)]$

(Th). Let A be a domain f.g /k. Then $\dim A = \text{tr.deg}\, Q(A)/k$.

Pf (β) + (γ)

~~[illegible]~~

* if $A \xrightarrow{u} B$ and $p \subset A$ prime we put $B_p := S^{-1}B$ where $S = u(A \setminus p)$.

(Th) (Hilbert's Nullstellensatz) k alg closed field, $\mathfrak{a} \subsetneq k[x_1, \dots, x_n]$ ideal $\Rightarrow \exists\, \alpha \in k^n$, $f(\alpha) = 0\ \forall f \in \mathfrak{a}$

[illegible]

Lecture 3 Noetherian rings. ~~[illegible]~~ (Dedekind) Discrete valuation rings.

(Def) An A-module M is called Noetherian if $\forall$ ascending chain of submod is stationary ($M_1 \subseteq M_2 \subseteq \cdots \Rightarrow \exists i$ s.t. $M_i = M_j\ \forall j > i$). A ring is called Noeth if ...

(α) (Prop) M is noeth iff any submod is fin gen [Pf: ...]

E (Prop) $0 \to M' \to M \to M'' \to 0$ exact (i.e. $M'' = M/M'$). Then M Noeth $\Leftrightarrow$ M' & M'' Noeth

(β) (Cor) A Noeth ring, M fin gen A-mod $\Rightarrow$ M Noeth [Pf: $M = A^n/N$]

(Th) (Hilbert basis th.) A Noeth $\Rightarrow$ $A[x_1, \dots, x_N]$ Noeth.

Pf. Ind $\Rightarrow$ suff for $n=1$. Put $B = A[X]$. Let $\underline{b} \subset B$ ideal. Let $\underline{a} = \{ a \in A \mid \exists f \in \underline{b}$ whose leading coeff is $a\}$ i.e. $f = ax^n + \dots \}$. $\Rightarrow \underline{a} = (a_1, \dots, a_r) \Rightarrow \exists f_i = a_i x^{k_i} + \dots \in \underline{b}$

Let $k = \max k_i$, $\underline{b}' = (f_1, \dots, f_r) \subset \underline{b}$. Let $M = A + Ax + \dots + Ax^{k-1} \subset B$.

Claim: $\underline{b} = (\underline{b} \cap M) + \underline{b}'$ (this closes the pf for $\underline{b} \cap M$ is f.g. A-mod by (α) & (β)) (hence as B-mod)

$\supset$ clear; for $\subset$ induction on degrees. Let $f \in \underline{b} \Rightarrow f = ax^n + \dots \Rightarrow a = \sum u_i a_i$

$\Rightarrow f - \sum u_i f_i x^{n-k_i} \in \underline{b}$ and has degree $< n \overset{\text{ind}}{\Rightarrow} f - \underbrace{\sum u_i f_i x^{n-k_i}}_{\in \underline{b}'} \in (\underline{b} \cap M) + \underline{b}' \Rightarrow f \in (\underline{b} \cap M) + \underline{b}'$.

(Cor) $\begin{cases} A \text{ Noeth}, B \text{ is fin gen } A\text{-alg} \Rightarrow B \text{ Noeth.} \\ k[x_1, \dots, x_N] \text{ Noeth.} \end{cases}$

E (Prop) A Noeth $\Rightarrow S^{-1}A$ Noeth [use $b = S^{-1}(b^c)$]

(Prop) Let A be Noeth. Any rad ideal a (i.e. $a = \sqrt{a}$) can be uniquely written as $p_1 \cap \dots \cap p_r$, p_i primes. [p_i are called the components of a].

Pf: Call an ideal a irred if $a = b \cap c \Rightarrow a = b$ or $a = c$.

Claim 1: any ideal = finite inters of irred ones [Let Σ be the set of ideals which are not finite inters of irr id; if $\Sigma \neq \emptyset$ it has a max elt (by Noeth) a $\Rightarrow a = b \cap c$, $a \subsetneq b$, $a \subsetneq c \Rightarrow b \notin \Sigma, c \notin \Sigma \Rightarrow a \notin \Sigma$ contradiction]. Claim 2. Any irred radical ideal a is prime [passing to A/a can ass $a = (0)$. So let (0) be irred $xy = 0$ but $x \neq 0, y \neq 0$. Let $\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2) \subseteq \cdots \Rightarrow \mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1}) \Rightarrow (x^n) \cap (y) = 0$ (check!)

E $\Rightarrow (x^n) = 0 \Rightarrow x = 0$ contradiction] Uniqueness easy.

(Prop) A Noeth ((0) reduced) $a \subset A$ id $\Rightarrow \exists n$ $(\sqrt{a})^n \subset a$.

(Def) K a field. A discrete val is a (surj) map $v : K \to \mathbb{Z} \cup \{\infty\}$ s.t. $\begin{cases} v(x) = \infty \Leftrightarrow x = 0 \\ v(xy) = v(x) + v(y) \\ v(x+y) \geq \min(v(x), v(y)) \end{cases}$

(Ex) $K = \mathbb{Q}$, v = exponent of p in x. ; $K = k((x))$, v = order

E (Prop) If $v : K \to \mathbb{Z} \cup \{\infty\}$ is a discr val $\Rightarrow A = \{ x \in K \mid v(x) \geq 0 \}$ is a local ring with max ideal $m = \{ x \in K \mid v(x) > 0 \}$. Moreover if $v(x) = 1 \Rightarrow$ any ideal has the form (x^n) ($\Rightarrow A$ is Noeth).

Pf: $v(1) = 0$, If $v(x) \geq 0 \Rightarrow 0 \leq v(x^2) = v((-x)^2) = 2v(-x) \Rightarrow v(-x) \geq 0 \Rightarrow A$ subring easy.

(Def) A domain A is called DVR if $\exists$ val $v : Q(A) \to \mathbb{Z} \cup \{\infty\}$ s.t. $A = \{ x \in K \mid v(x) \geq 0 \}$.

So a DVR is local, Noeth of dim = 1. Any $x \in A$ s.t. $v(x) = 1$ is called a parameter of A.

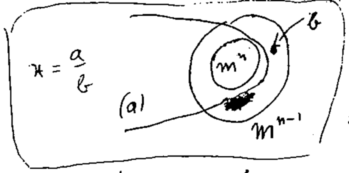
(Th) A Noeth domain, A local, $\dim A = 1$, m its max id, $k = A/m$ (res field). Equiv:

1) A is DVR
2) A is integrally closed
3) m principal ($m = (x)$)
4) $\dim_k m/m^2 = 1$
5) $m = (x)$ & $\forall$ non-zero ideal has form (x^k).

Pf. 1)⇒2) Let $x \in Q(A)$, $x^n + a_1 x^{n-1} + \dots + a_n = 0$, $a_i \in A$. If ~~...~~ $v(x) \geq 0$ O.K. If $v(x) < 0$ ⇒

⇒ $v(x^{-1}) > 0$ ⇒ $x^{-1} \in A$ $\overset{: x^{n-1}}{\Rightarrow}$ ~~...~~ $\underbrace{x + a_1 + a_2 x^{-1} + \dots + a_n x^{-n+1}}$ $= 0$ ⇒ $x \in A$ abs.

2)⇒3) Let $0 \neq a \in m$. ⇒ $\sqrt{(a)} = m$ (claim $m = (x)$) ⇒ ~~...~~ $\overset{A}{\exists n}$ s.t. $m^n \subset (a)$, $m^{n-1} \not\subset (a)$. Choose $b \in m^{n-1} \setminus (a)$, $x = a/b \in Q(A)$. We have $x^{-1} \notin A$ (for $b \notin (a)$) ⇒ x^{-1} not



integral $/A$ so $x^{-1} m \not\subset m$ (for m is a faithfull $A[x^{-1}]$-module f.g. $/A$ (Theorem)

But $x^{-1} m \subset A$ (for $x^{-1} m = \frac{b}{a} m$ and $bm \subset m^n \subset (a)$) so

$x^{-1} m = A$ so $m = xA = (x)$.

3)⇔4) ~~...~~ by Nakayama. (lect 2)

3)⇒5) Let $\mathfrak{a} \subset A$ ideal $\neq (0), (1)$. ⇒ $\sqrt{\mathfrak{a}} = m \overset{(x)}{=}$ ⇒ $\exists n$ s.t. $(x^n) \subset$ ~~...~~

~~...~~ $\overset{Nak}{\Rightarrow} \exists r$ s.t. $\mathfrak{a} \subset (x^r)$, $\mathfrak{a} \not\subset (x^{r+1})$

Choose $y \in \mathfrak{a} \setminus (x^{r+1})$ and write $y = \alpha x^r$ ($\alpha \in A$). ⇒ $\alpha \notin m = (x)$

⇒ α invertible ⇒ $x^r \in \mathfrak{a}$ ⇒ ~~...~~ ⇒ $\mathfrak{a} = (x^r)$.

5)⇒1) $(x^k) \neq (x^{k+1})$ by Nakayama so $\forall y \in A$, $(y) = (x^k)$ for exactly one k. Put $v(y) = k$ and $v(y/z) = v(y) - v(z)$. Check it is OK.

Def A Dedekind domain is a Noeth domain A integrally closed of $\dim A = 1$.

Prop Let A be Noeth (domain) of $\dim A = 1$. Equiv are:

1) A is integrally closed (i.e. A is Dedekind)
2) $\forall\, p \neq 0$ prime, A_p is a DVR. [Pf. Lect 2, (δ) and ~~...~~ above Th.]

Prop Let A be DVR with parameter x. Then if $k = A/(x)$ $\dim_k \frac{A}{(x^n)} = n$.
$k \hookrightarrow A \to k$ is id

Pf. Ex. seq. $0 \to \frac{(x^n)}{(x^{n+1})} \to \frac{A}{(x^{n+1})} \to \frac{A}{(x^n)} \to 0$. So suff $\dim \frac{(x^n)}{(x^{n+1})} = 1$. That ≤ 1 clear. That $\neq 0$ of Nakayama.

Ex $f \in k[x,y]$ irred polyn $f(0,0) = 0$ (i.e. $f \in (x,y)$) and $A = (k[x,y]/(f))_{(\bar{x},\bar{y})}$

Then A is a local domain of dim 1 (Lect 2, (α)). It is a DVR iff either $a := \frac{\partial f}{\partial x}(0,0) \neq 0$ or $b := \frac{\partial f}{\partial x}(0,0) \neq 0$. [For if $f = ax + by +$ (higher terms) then $\frac{m}{m^2} \simeq \frac{kx + ky}{k(ax + by)}$ because $m = (x,y)/(f)$, $m^2 = ((x,y)^2 + (f))/(f)$ so $\frac{m}{m^2} = \frac{(x,y)}{(x,y)^2 + (f)} = \frac{(x,y)}{(x,y)^2 + (ax+by)} = \frac{(x,y)/(x,y)^2}{((x,y)^2 + (ax+by))/(x,y)^2}$

So if $a \neq 0$, $\bar{y}$ is a par while if $b \neq 0$, $\bar{x}$ is a par.

Ex $\underbrace{A = (k[x,y]/(y^2 - x^3))_{(\bar{x},\bar{y})}}_{\text{not a DVR.}} \xrightarrow[\substack{x \mapsto t^2 \\ y \mapsto t^3}]{\sim} k[t^2, t^3]_{(t^2,t^3)} \hookrightarrow k[t]_{(t^2,t^3)} =$ integral clos of A

~~(**) A Noeth, $\mathfrak{a} \subset A$ ideal ⇒ $\exists n$ s.t. $(\sqrt{\mathfrak{a}})^n \subset \mathfrak{a}$. Pf: ...~~

cf Lect 2

~~(*) Lemma M f.g. A-mod, A local with max id m. Ass $x_1, \dots, x_n \in M$ are s.t. $\bar{x}_1, \dots, \bar{x}_n$ gen M/mM as a $k = A/m$-vect sp. Then $x_1, \dots, x_n$ gen M.~~

~~Pf. $M' = \sum A x_i$. Then $m(M/M') = \frac{mM + M'}{\dots} = \frac{M}{\dots}$ so $M/M' = 0$ so $M = M'$~~

Lecture 4. ~~Affine~~ Algebraic ~~varieties~~ sets in $\mathbb{A}^n$ and $\mathbb{P}^n$

Def k = alg cl field (e.g. $k=\mathbb{C}$), $\mathbb{A}^n = k^n$ (affine space), $P=(\alpha_1,\dots,\alpha_n)\in\mathbb{A}^n$ (point ↓, coordinates ↓)

For a subset $T\subset A = k[x_1,\dots,x_n]$ put $Z(T)=\{P\in\mathbb{A}^n \mid f(P)=0, \forall P\in T\}$ (the zeroes of T)

If $a=(T) \Rightarrow Z(T)=Z(a)=Z(\sqrt{a})$.

Def A subset $Y\subset\mathbb{A}^n$ is called algebraic set if $\exists T$ s.t. $Y=Z(T)$

$\in$ **Prop** Alg sets are the closed sets of a top on $\mathbb{A}^n$ (called Zar top). [$Z(T_1T_2)=Z(T_1)\cup Z(T_2)$; $Z(1)=\emptyset$; $Z(0)=\mathbb{A}^n$; $Z(\cup T_\alpha)=\cap Z(T_\alpha)$]

Ex for $\mathbb{A}^1$: the finite sets are all. **Ex** Linear subsets (= vect. subsp.) { lines, hyperplanes

Def For a subset $Y\subset\mathbb{A}^n$ define $I(Y)=\{f\in A \mid f(P)=0 \text{ for all } P\in Y\}$ (the ideal of Y)

Prop (Hilb Nullstellensatz) For any ideal $\underline{a}\subset A$ we have $I(Z(\underline{a}))=\sqrt{\underline{a}}$

Pf. Let $f(P)=0$ for all $P\in Z(\underline{a})$. Ass ~~[...]~~ $f\notin\sqrt{\underline{a}}$. ~~[...]~~

$\Rightarrow \exists\, \underline{p}\subset A$ prime $f\notin\underline{p}$, $\underline{a}\subset\underline{p}$. Let m be a max id in $(A/\underline{p})_f$; the field $L=(A/\underline{p})_f/m$ is f.g/k as an alg. Noether norm $\Rightarrow$ L is integral over some $k[y_1,\dots,y_d]$

$\Rightarrow k[y_1,\dots,y_d]$ field $\Rightarrow d=0 \Rightarrow L/k$ algebraic $\Rightarrow L=k$. If $\bar{x}_i\in L=k$ image of x_i

$\Rightarrow f(\bar{x}_1,\dots,\bar{x}_n)\neq 0$ but $(\bar{x}_1,\dots,\bar{x}_n)\in Z(\underline{p})\subset Z(\underline{a})$ contr.

$\in$ **Prop** For any subset $Y\subset\mathbb{A}^n$, $Z(I(Y))=\overline{Y}$ (Zar clos of Y).

Cor $Y\mapsto I(Y)$ & $\underline{a}\mapsto Z(\underline{a})$ give a bij bet alg subsets of $\mathbb{A}^n$ & rad ideals of A.

Under this corresp. pts corr to max ideals. Moreover $Z(\underline{a}\cap\underline{b})=Z(\underline{a})\cup Z(\underline{b})$, $I(Y_1\cup Y_2)=I(Y_1)\cap I(Y_2)$

Def A ~~[...]~~ top sp X is called irred if $X=X_1\cup X_2$, X_i closed $\Rightarrow X=X_1$ or $X=X_2$.

A subset $Y\subset X$ is called irr if it is so as a top sp (with induced top)

An open set of an irr sp is dense & irr.

Prop An alg set $Y\subset\mathbb{A}^n$ is irred iff $I(Y)$ is prime [Use: a rad id is irred iff it is prime]

Any alg set $Y\subset\mathbb{A}^n$ can be written uniquely $Y=\bigcup_{i=1}^m Y_i$, Y_i irr, $Y_i\not\subset Y_j$ [Use the stat for rad id]

~~[...]~~

Def X a top space. Define $\dim X$ as the max n s.t. $\exists\ Y_0\supsetneq Y_1\supsetneq\dots\supsetneq Y_n$, Y_i irr.

Def For $Y\subset\mathbb{A}^n$ alg set. Put $A(Y)=A/I(Y)$ (= coordinate ring) Rem Max $A(Y)\overset{\text{bij}}{\simeq} Y$ (Spec $A(Y)$ = irr closed subsets)

Prop $\dim Y=\dim A(Y)$ in $\S$. So $\dim\mathbb{A}^n=n$.

~~**Def** By an affine curve we mean an irr aff alg set of dim 1. We call it non-sing if $A(Y)$ is integr closed (so Dedekind!!)~~

~~**Ex** If $f\in k[x,y]$ irred s.t. $f=\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$ has no sol in $\mathbb{A}^2$. Then $Z(f)\subset\mathbb{A}^2$ is a non-sing curve~~

~~non-sing: e.g. $f=y^2-F(x)$, F without multiple roots~~

~~or $f=y^n+x^n-1$~~

~~$y^2=(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$, $\lambda_1\neq\lambda_2\neq\lambda_3\neq\lambda_1$~~

~~← reducible~~

~~**Def** A quasi-aff alg set is an open set of an aff alg set. A quasi-aff curve is ...~~

~~[...] in $\mathbb{R}^2$ in the usual top!~~

./.

(Def) $\mathbb{P}^n := k^{n+1} \setminus \{0\} / \sim$ where $(a_0, \dots, a_n) \sim (a_0', \dots, a_n')$ if $\exists \lambda \in k^*$, $a_i' = \lambda a_i \ \forall i$.

$[a_0 : \dots : a_n]$ the eq class $\in \mathbb{P}^n$. (a_i = hom coord); $S = k[x_0, \dots, x_n]$, it is graded.

E (Def) An ideal $\mathfrak{a} \subset S = k[x_0, \dots, x_n]$ is called hom if $f \in \mathfrak{a} \Rightarrow$ the hom comp $f_d \in \mathfrak{a}$. Call S_d = hom pol of deg d and $S^h = \cup S_d$. We have $S = \oplus S_d$. ($f = \Sigma f_d$..)

(Prop) $+, \cdot, \cap, \sqrt{\ }$ of hom id is hom!

(Def) For $T \subset S^h$ put $Z(T) = \{ P \in \mathbb{P}^n \mid f(P) = 0 \text{ for all } f \in T \}$

If $\mathfrak{a} \subset S$ is homog. define $Z(\mathfrak{a}) = Z(\mathfrak{a} \cap S^h)$. By an alg set we mean one of the form $Z(T)$.

(Ex) Linear subsets, lines, hyperplanes.

E (Prop) Alg sets are the cl. sets of a top (Zariski) $[Z(S_+) = \emptyset]$.

(Def) For $Y \subset \mathbb{P}^n$ alg set, $I(Y)$ = ideal gen by $\{ f \in S^h \mid f(P) = 0, \forall P \in Y \}$

$S(Y) = S / I(Y)$ = homog. coord. ring.

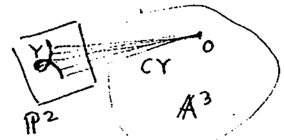
E (Prop) $\mathfrak{a} \mapsto Z(\mathfrak{a})$ & $Y \mapsto I(Y)$ give a bij bet alg sets in $\mathbb{P}^n$ and hom. rad ideals in S (different from S_+)

Y irr $\Leftrightarrow$ $I(Y)$ prime [use the affine version]

(Def) Lin subspaces, hyperplanes, lines, hypersurfaces

(Def) $Y \subset \mathbb{P}^n$ alg set, $\theta : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, $\theta(a_0, \dots, a_n) = [a_0 : \dots : a_n]$. Define the affine cone

$C(Y) = \theta^{-1}(Y) \cup \{0\} \subset \mathbb{A}^{n+1}$. Clearly $C(Y)$ aff alg set & $I(C(Y)) = I(Y)$.



Ex. $Y = Z(x^2 z - y^3) \subset \mathbb{P}^2$

$CY = Z(x^2 z - y^3) \subset \mathbb{A}^3$

(Def) Let $H_i = Z(x_i) \subset \mathbb{P}^n$ $(0 \le i \le n)$ and $\varphi_i : U_i = \mathbb{P}^n \setminus H_i \to \mathbb{A}^n$, $\varphi_i(a_0, \dots, a_n) = \left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)$ with $\frac{a_i}{a_i}$ ommited.

(Prop) φ_i are homeo [Pf: bij is clear. Put $U = U_0$, $\varphi = \varphi_0$, $S = k[x_0, \dots, x_n]$, $A = k[y_1, \dots, y_n]$.

Define maps $\alpha : S^h \to A$ (deshomogeneisation), $\beta : A \to S^h$ by $\alpha(f) = f(1, y_1, \dots, y_n)$ and (for $g \in A$, $\deg(g) = e$), $\beta(g) = x_0^e \, g(x_1/x_0, \dots, x_n/x_0)$ ✱. Let $Y \subset U$ be closed, $\bar{Y}$ its closure in $\mathbb{P}^n$ so $\bar{Y} = Z(T)$ for some $T \subset S^h$. Let $T' = \alpha(T)$

E We check that $\varphi(Y) = Z(T')$. Conversely if $W \subset \mathbb{A}^n$ closed, $W = Z(T')$ for $T' \subset A$ $\left(= k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right]\right)$

$\Rightarrow \varphi^{-1}(W) = Z(\beta(T')) \cap U$. So φ, φ^{-1} are closed maps, O.K.

E (Remarks) 1) $I(\bar{Y}) = \sqrt{\text{ideal}}$ of S generated by $\beta(I(Y))$ for any $Y \subset \mathbb{A}^n$ closed (where $\bar{Y} \subset \mathbb{P}^n$ closure)

2) If $Z \subset \mathbb{P}^n$ closed $\Rightarrow Z \cap \mathbb{A}^n = Z(\alpha(I(Z) \cap S^h))$. If identify $k[y_1, \dots, y_n]$ with $k[x_0, \dots, x_n]_{(x_0)}$ then $I(Z \cap \mathbb{A}^n) = I(Z)_{(x_0)}$

(Ex) The clos of $Z(y^2 - x - x^3) \subset \mathbb{A}^2$ in $\mathbb{P}^2$ has the ideal gen by $y^2 z - xz^2 - x^3$

(Rem) X top sp, $U \subset X$ open, $Y \subset U$, Y closed in U, $\bar{Y}$ = clos of Y in $X \Rightarrow \bar{Y} \cap U = Y$

Conversely if $T \subset X$ is closed irred and $T \cap U \neq \emptyset \Rightarrow \overline{T \cap U} = T$.

Pf: $Y = Z \cap U$, Z closed in $X \Rightarrow Y \subset Z \Rightarrow \bar{Y} \subset Z \Rightarrow \bar{Y} \cap U \subset Z \cap U = Y$ q.e.d.

(Cor) $\mathbb{P}^n$ is Noeth. of dim. n.

(Cor) X top sp, $U \subset X$ open $\Rightarrow \dim U \le \dim X$ [If $Y_0 \subsetneq Y_1 \subsetneq \cdots$ then by the above rem $\bar{Y}_0 \subsetneq \bar{Y}_1 \subsetneq \cdots$]

✱✱ If $n = 3$ $f = x_1^2 x_0 - x_2^3 \Rightarrow \alpha(f) = y_1^2 - y_2^3$

$g = y_1^2 - y_2^3 \Rightarrow \beta(g) = x_1^2 x_0 - x_2^3$

$f = x_0 \Rightarrow \alpha(f) = 1$ $\boxed{\alpha, \beta \text{ not inverse!}}$

$f = x_0^2 - x_0 x_1 \Rightarrow \alpha(f) = 1 - y_1 \Rightarrow \beta(\alpha(f)) = x_0 - x_1$

✱ $f(P)$ does not make sense for $f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$; but "$f(P) = 0$" makes sense

Lecture 5. Algebraic sets in $\mathbb{A}^n \times \mathbb{P}^m$ and $\mathbb{P}^n \times \mathbb{P}^m$

Def Let $R = k[y_1,\dots,y_n,x_0,\dots,x_m] = \oplus R_d$ (R_d = polyn which are hom of deg d in $x_0,\dots,x_m$) (it is graded ring with $R_0 = k[y_1,\dots,y_n]$). For $T \subset R^h$ put $Z(T) = \{(P,Q) \in \mathbb{A}^n \times \mathbb{P}^m \mid f(P,Q) = 0$ for all $f \in T\}$; for $\underline{a} \subset R$ hom def $Z(\underline{a}) = Z(\underline{a} \cap R^h)$. By an alg set $Y \subset \mathbb{A}^n \times \mathbb{P}^m$ we mean $Z(T)$ for some T. For $Y \subset \mathbb{A}^n \times \mathbb{P}^m$ alg set put $I(Y) =$ = id gen in R by $\{f \in R^h \mid f(P,Q) = 0, \forall (P,Q) \in Y\}$ and $R(Y) = R/I(Y)$.

≡ **Prop** $\underline{a} \mapsto Z(\underline{a})$, $Y \mapsto I(Y)$ give bij bet. alg sets in $\mathbb{A}^n \times \mathbb{P}^m$ and hom rad id in R s.t. $\sqrt{\underline{a}} \not\supset R_+$.

≡ **Prop** $1 \times \varphi_i : \mathbb{A}^n \times (\mathbb{P}^m \setminus H_i) \to \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ are homeo. (φ_i as in Lect. 4)(**)

Th (elimin. theory). Let $Y \subset \mathbb{A}^n \times \mathbb{P}^m$ be alg set and $f : \mathbb{A}^n \times \mathbb{P}^m \to \mathbb{A}^n$ the 1st proj. Then $f(Y)$ is an alg set (*)

Pf. Let $I = I(Y) \subset R = k[y,x]$, $A = k[y]$, $U_i = \mathbb{P}^m \setminus H_i$. Then $I(Y \cap (\mathbb{A}^n \times U_i)) =$ $= I_{(x_i)} \subset R_{(x_i)}$. Let $P \in \mathbb{A}^n \setminus f(Y)$. We prove $\exists$ open nbhd V of P in $\mathbb{A}^n$ s.t. $V \cap f(Z) = \phi$. Let $\underline{m} = m_P \subset A$ be the max id corr to P; then no max id in $R_{(x_i)}$ containing $I_{(x_i)}$ lies over m (***). So $\underline{m} R_{(x_i)} + I_{(x_i)} = R_{(x_i)}$, so $1 = \sum_j m_{ij} w_{ij} + a_i$, $m_{ij} \in \underline{m}$, $w_{ij} \in R_{(x_i)}$, $a_i \in I_{(x_i)}$. So $x_i^{N_i} = \sum m_{ij} w'_{ij} + a'_i$, $w'_{ij} \in R$, $a'_i \in I$. Taking hom comp of deg N_i we may ass $w'_{ij} \in R_{N_i}$, $a'_i \in I_{N_i}$. Choose $N \geq \sum N_i \Rightarrow$ $R_N \subset mR_N + I_N \subset R_N \Rightarrow$ $m(R_N/I_N)_m = (R_N/I_N)_m$. Nak $\Rightarrow (R_N/I_N)_m = 0 \Rightarrow \exists g \in A \setminus m$ s.t. $g R_N \subset I_N$ $\Rightarrow g x_i^N \in I_N \Rightarrow g \in I_{(x_i)} \Rightarrow \mathbb{A}^n \setminus Z(g) \subset \mathbb{A}^n \setminus f(Y)$, q.e.d.

Def Let $B = k[u_0,\dots,u_n,v_0,\dots,v_m] = \bigoplus_{d,e} B_{de}$ [$B_{d,e}$ = polyn which are hom in both u and v of degree d,e], $B^h = \cup B_{d,e}$ (4). For $T \subset B^h$ put $Z(T) = \dots$ (anal) $\subset \mathbb{P}^n \times \mathbb{P}^m$. An ideal is called hom if it is gen by elts in B^h. Define $Z(\underline{a})$ analogously for $\underline{a}$ hom. An alg set $Y \subset \mathbb{P}^n \times \mathbb{P}^m$ is a set of the form $Z(T)$. For such an Y put $I(Y) = \dots$ (as above) and $B(Y) = B/I(Y)$.

≡ **Prop** $\varphi_i \times \varphi_j : (\mathbb{P}^n \setminus H_i) \times (\mathbb{P}^m \setminus H_j) \to \mathbb{A}^n \times \mathbb{A}^m$ homeo.

(4) So called $\mathbb{Z} \times \mathbb{Z}$-graded ring

(***) Max ideals in $R_{(x_i)} = k[y_1,\dots,y_n,z_1,\dots,z_m]$ corr to $\mathbb{A}^{n+m}$, max id of A corr to $\mathbb{A}^n$ and lying over corr (obviously to the projection on 1st factor). Max id of $R_{(x_i)}$ containing $I_{(x_i)}$ corr to pts of $Y \cap (\mathbb{A}^n \times U_i)$.

(**) Moreover if $Y \subset \mathbb{A}^n \times \mathbb{P}^m$ alg set & we identify $\underbrace{k[y_1,\dots,y_n,x_0,\dots,x_m]_{(x_0)}}_{k[y_1,\dots,y_n,\frac{x_1}{x_0},\dots,\frac{x_m}{x_0}]}$ with $k[y_1,\dots,y_n,z_1,\dots,z_m] = A(\mathbb{A}^n \times \mathbb{A}^m)$ then $I(Y \cap (\mathbb{A}^n \times (\mathbb{P}^m \setminus H_0))) = I(Y)_{(x_0)}$.

(*) This fails for 1st proj $\mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^n$; e.g. $Y = Z(xy-1) \subset \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ projects onto $\mathbb{A}^1 \setminus \{0\}$

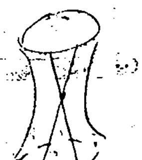
(Prop) ~~Construction~~ Let $s: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ be given by $s(u_0:\dots:u_n; v_0:\dots:v_m) = (\dots : u_i v_j : \dots)$ (called Segre map). Then $X = \operatorname{Im} s$ is closed and $s: \mathbb{P}^n \times \mathbb{P}^m \to X$ is a homeo.

Pf. Claim: $\operatorname{Im} s = \{(\dots : w_{ij} : \dots) \mid w_{ij} w_{kl} = w_{kj} w_{il}\} = \{(\dots : w_{ij} : \dots) \mid \operatorname{rank} \begin{vmatrix} w_{00} & \cdots & w_{0m} \\ \vdots & & \\ w_{n0} & \cdots & w_{nm} \end{vmatrix} = 1\}$
["$\subset$" clear; "$\supset$" if rank $=1$ all rows prop. to one of them]. s is injective (by direct verif.). Let's prove s, s^{-1} closed maps. If $F \in B_{d,e}$ and say $e \geq d$ then put $G_i = u_i^{e-d} F$. The G_i's have degree e in both u_i and v_j; $\overset{(*)}{\Rightarrow}$ $\exists$ polyn $\tilde{G}_i \in R = k[w_{ij}]$ s.t. $G_i(u_0, \dots, u_n, v_0, \dots, v_m) = \tilde{G}_i(\dots, u_i v_j, \dots)$. Then s takes $Z(F)$ into $Z(\tilde{G}_0, \dots, \tilde{G}_n)_{\cap X}$. Conversely if $H \in R$ put $\underset{\sim}{H}(u_0, \dots, u_n, v_0, \dots, v_m) := H(\dots, u_i v_j, \dots)$. Then s^{-1} takes $Z(H)_{\cap X}$ into $Z(\underset{\sim}{H})$.

Ex $s: \underset{(x,z)}{\mathbb{P}^1} \times \underset{(y,w)}{\mathbb{P}^1} \to \underset{(x_0 x_1, x_2, x_3)}{\mathbb{P}^3}$ has the image $Z(xy - zw) = Z(t_0^2 + t_1^2 + t_2^2 + t_3^2)$

E $s(\mathbb{P}^1 \times \{P\})$ & $s(\{Q\} \times \mathbb{P}^1)$ are straight lines$^{(**)}$ in $\mathbb{P}^3$

quadric



E.g. $\mathbb{P}^1 \times (1:0) \mapsto Z(x_2, x_4)$

$\mathbb{P}^1 \times (0:1) \mapsto Z(x_1, x_3)$

PP. $X \times Y = Z_1 \cup Z_2$. For $P \in X \Rightarrow P \times Y \subset Z_1$ or $P \times Y \subset Z_2$. Suf $X_1 = \{P \in X \mid P \times Y \subset Z_1\}$ is closed. But $X_1 = \cap X_Q$ where $X_Q = \{P \in X \mid (P,Q) \in Z_1\}$ is closed.

Examples of alg subsets of products

Ex $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ alg sets $\Rightarrow X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ alg set. Do X, Y irred $\Rightarrow X \times Y$ irreduct.
$X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ — " — $X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$ alg set

Ex 1) $X \subset \mathbb{A}^n$ alg set. $\Rightarrow$ the diag $\Delta_X \subset X \times X$ is closed [$\Delta_X = (X \times X) \cap \Delta_{\mathbb{A}^n}$ and $\Delta_{\mathbb{A}^n} = Z(y_1 - z_1, \dots, y_n - z_n)$ if $A(\mathbb{A}^n) = k[y_1, \dots, y_n]$, $A(\mathbb{A}^n) = k[z_1, \dots, z_n]$]
2) $X \subset \mathbb{P}^n$ alg set $\Rightarrow \Delta_X \subset X \times X$ closed [$\Delta_{\mathbb{P}^n} = Z(\dots, u_i v_j - u_j v_i, \dots)$]

Ex Think $\mathbb{P}^n$ = set of lines $L \subset \mathbb{A}^{n+1}$ (line = 1-dim lin subspace). Then $X = \{(P, L) \in \mathbb{A}^{n+1} \times \mathbb{P}^n \mid P \in L\} \subset \mathbb{A}^{n+1} \times \mathbb{P}^n$ is closed ($= Z(\dots, x_i y_j - x_j y_i, \dots)$)

E Analyze the projections $X \to \mathbb{A}^{n+1}$, $X \to \mathbb{P}^n$ (as usual x_i coord in $\mathbb{P}^n$, y_j coord in $\mathbb{A}^n$)

Ex ~~Def~~ By a hyperplane $H \subset \mathbb{P}^n$ we understand an alg set $Z(L)$ where $L \in S_1$, $L \neq 0$. The set of hyperplanes (called dual proj space $(\mathbb{P}^n)^*$) is isom to a proj space $\mathbb{P}^n$. Then $X = \{(P, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^* \mid P \in H\}$ is closed ($X = Z(v_0 u_0 + \dots + v_n u_n) \subset \mathbb{P}^n \times \mathbb{P}^n$).

Ex $X \subset \mathbb{P}^n$ alg set. Then $\Gamma \subset [(X \times X) \setminus \Delta_X] \times \mathbb{P}^n$, $\Gamma := \{(P, Q, R) \mid R \in \overline{PQ}\}$ is closed [condition $R \in \overline{PQ}$ is written $\operatorname{rank} \begin{pmatrix} x_0 & \cdots & x_n \\ u_0 & \cdots & u_n \\ v_0 & \cdots & v_n \end{pmatrix} \leq 2$]. Def: the closure of the image of Γ in $\mathbb{P}^n$ is $\operatorname{Sec}(X)$ (Secant variety).

(**) an alg set $\subset \mathbb{P}^n$ is called a lin subvar if it is given by $Z(L_1, \dots, L_r)$, $\deg L_i = 1$. It is called hyperpl if $r = 1$. It is called a line if $r = n-1$.

(*) any pol in $B_{e,e}$ is a polyn ~~in~~ in the $u_i v_j$'s of degree e

(0) Topol induced on $P \times Y$ from $X \times Y$ coincides with top on Y ind

Lecture 6. Algebraic varieties

Def Let X be a top. sp, k a field. By a sheaf $\mathcal{O}$ of rings of fns on X we mean the giving for each open $U \subset X$ of subring $\mathcal{O}(U)$ of the ring of all fns $: X \to k$ s.t. whenever $U \subset X$ open, $U = \cup U_i$ open covering we have ($f \in \mathcal{O}(U) \Leftrightarrow f/_{U_i} \in \mathcal{O}(U_i)\ \forall i$) for $f : U \to k$ (***)
For $P \in X$ put $\mathcal{O}_P = \{(U, f) \mid U \ni P, f \in \mathcal{O}(U)\}/\sim$ (ring of germs).
Ex : $k = \mathbb{R}$, $\mathcal{O}$ = cont ~~maps~~ fns.

Def (categ of ringed spaces(+)) Objects : pairs $(X, \mathcal{O}_X)$, X top sp, $\mathcal{O}_X$ a sheaf of rings of fns. A morph $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ (written simply $f : X \to Y$) is a cont map $f : X \to Y$ s.t. $\forall V \subset Y$ open $\forall \varphi \in \mathcal{O}_Y(V) \Rightarrow (f^{-1}(V) \xrightarrow{f} V \xrightarrow{\varphi} k) \in \mathcal{O}_X(f^{-1}(V))$. Any open subset of X has a str of ringed sp.

Def Let Y open in some alg set X in $\mathbb{A}^n$ (or $\mathbb{P}^n$ or $\mathbb{A}^n \times \mathbb{P}^m$, or $\mathbb{P}^n \times \mathbb{P}^m$). Define a sheaf $\mathcal{O}_Y$ as foll. A fn $f : V \to k$ ($V \subset Y$ open) is called reg at $P \in V$ if $\exists\, W \subset V$ open ($W \ni P$) and $g, h \in A$ (or S_d or R_d or $B_{d,e}$) s.t. $h(Q) \neq 0$, $\forall Q \in W$ (**) & $f(Q) = \frac{g(Q)}{h(Q)}$, $\forall Q \in W$. Put $\mathcal{O}_Y(V) = \{$fns $: V \to k$ reg at any $P \in V\}$. Clearly a sheaf (& check elts of $\mathcal{O}$ are cont!)

E **Prop** $\varphi_i : \mathbb{P}^n \setminus H_i \to \mathbb{A}^n$ in Lecture 4 are iso of ringed sp. So are $1 \times \varphi_i$, $\varphi_i \times \varphi_j$ a.s.o.

Prop Let $X \subset \mathbb{A}^n$ be an alg set ($X = Z(f_1, \dots, f_m)$) & $f \in A = k[x_1, \dots, x_n]$. Then $X_f := X \setminus Z(f)$ is isomorphic (as a ringed sp.) with $Y = Z(f_1, \dots, f_m, x_{n+1}f - 1) \subset \mathbb{A}^{n+1}$.

E Pf. Let $u : X_f \to Y$ be given by $u(a_1, \dots, a_n) = (a_1, \dots, a_n, \frac{1}{f(a_1, \dots, a_n)})$. Check ok.

Def (categ of varieties) Objects : ringed spaces X (irred as top sp) s.t. $\forall P \in X$ $\exists$ open nbhd U s.t. U isom (as a ringed sp) with $(Y, \mathcal{O}_Y)$ where $Y \subset \mathbb{A}^n$ alg set. Morph. are those of r. sp.
Rem Any open & any irr closed subset of a var has a str. of var.

Prop If Y is an open set of alg set in $\mathbb{A}^n$, $\mathbb{P}^n$, $\mathbb{A}^n \times \mathbb{P}^m$, $\mathbb{P}^n \times \mathbb{P}^m \Rightarrow (Y, \mathcal{O}_Y)$ variety.
Pf. If Y open in $X \subset \mathbb{A}^n$, $Y = Z(\underline{a}) \setminus Z(\underline{b})$ and $\underline{b} = (f_1, \dots, f_m) \Rightarrow Y = \bigcup_i (Z(\underline{a}) \setminus Z(f_i))$
If Y open in $X \subset \mathbb{P}^n$, $\mathbb{A}^n \times \mathbb{P}^n$, $\mathbb{P}^n \times \mathbb{P}^m$ use the φ_i's.

Def A var is called aff, q-aff, proj, q-proj (4) if it is isom with cl / open in cl in $\mathbb{A}^n$ / $\mathbb{P}^n$.
A var is called curve if it has dim = 1. A curve X is nonsing iff $\mathcal{O}_P$ is a DVR for all $P \in X$. (5)

Def Y var ; $K(Y) = \{(U, f) \mid U \subset Y, f \in \mathcal{O}(U)\}/\sim$ (***) It is a field called funct. field

E **Prop** If $s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ Segre $\Rightarrow s : \mathbb{P}^n \times \mathbb{P}^m \to s(\mathbb{P}^n \times \mathbb{P}^m)$ iso of var.

Cor Let X, Y be proj var ($X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$). Then $X \times Y$ (viewed as a var with its str of alg set in $\mathbb{P}^n \times \mathbb{P}^m$) is projective. [Pf : use Segre].

Cor Similar stat. with "q-proj" resp "affine" instead of "proj".

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(4) all 4 types are in partic q-proj!

(***) $(U, f) \sim (U', f')$ if $\exists\, U'' \subset U' \cap U$ s.t. $f/_{U''} = f'/_{U''}$.

(**) if $h \in R_d$, $h(Q)$ does not make sense but "$h(Q) = 0$" makes and so does $\frac{g(Q)}{h(Q)} \in k$.

(+) more restricted than in the literature!

(5) Ex : $f \in k[x,y]$ irr $X = Z(f) \subset \mathbb{A}^2$ aff curve ; $f, \partial f/\partial x, \partial f/\partial y$ have no common zero $\Leftrightarrow X$ non-sing (Lect 3), If $f = y^2 - P(x)$, X non-sing $\Leftrightarrow f$ simple root. $x^4 + y^4 + z^4$ gives non-sing.

if $f = y^2 - P(x)$ X non-sing iff P has distinct roots.

(Prop) Let $Y \subset \mathbb{A}^n$ alg set (as a var). Then $\mathcal{O}(Y) = A(Y)$, $K(Y) = Q(A(Y))$, $\mathcal{O}_P = A(Y)_{m_P}$ where $m_P = \{f \in A \mid f(P) = 0\} = I(\{P\})/I(Y)$. Moreover if $f \in \mathcal{O}(Y)$, $P \in Y$ then $f(P)$ is precisely the residue class of f modulo m_P viewed in $A(Y)/m_P \overset{can}{\simeq} k$.

Pf. $Y \simeq \operatorname{Max} A(Y)$ by Nullst. $\exists$ nat inj map $A(Y) \to \mathcal{O}(Y)$. By dfn of reg func $\alpha_P : A(Y)_{m_P} \to \mathcal{O}_P$ iso. $\Rightarrow Q(A(Y)) \simeq K(Y)$. Now $A(Y) \subset \mathcal{O}(Y) \subset \bigcap_{P \in Y} \mathcal{O}_P = \bigcap_{m \in \operatorname{Max}} A(Y)_m \overset{*}{=} A(Y)$

(α) (Prop) The functor $X \mapsto \mathcal{O}(X)$ is an equiv bet $\{$aff var$\}$ and $\{$integr dom. f.g. $/k\}$.

Pf. Clearly ess. surj. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ and check $\alpha : \operatorname{Hom}_{var}(X,Y) \to \operatorname{Hom}_{k\text{-alg}}(A(Y), A(X))$ iso [α induced by $\mathcal{O}(Y) \to \mathcal{O}(X)$]. We construct an inverse. Let $\bar{u} : k[y]/I(Y) \to k[x]/I(X)$. Lift it to $u : k[y] \to k[x]$, put $f_i = u(y_i)$. Consider $f = (f_1, \dots, f_m) : \mathbb{A}^n \to \mathbb{A}^m$. Then $f(X) \subset Y$ (if $P = (a_1, \dots, a_n) \in X$ we want $F(f_1(a_1,\dots,a_n), \dots, f_m(a_1,\dots,a_n)) = 0$ for $F \in I(Y)$. But $u(F(y_1,\dots,y_m)) = F(u(y_1), \dots, u(y_m)) = F(f_1(x_1,\dots,x_n), \dots, f_m(\ ))$. so $\uparrow = (u\circ F)(a_1,\dots,a_n) = 0$)

One checks $f : X \to Y$ morph

(Ex) $\mathbb{A}^1 \setminus \{0\} \neq \mathbb{A}^1$, $\mathbb{A}^2 \supset Z(xy-1) \simeq \mathbb{A}^1 \setminus \{0\}$, $X = \mathbb{A}^2 \setminus \{(0,0)\}$ is not affine [if it was aff, the incl $X \hookrightarrow \mathbb{A}^2$ being not iso $\Rightarrow \mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(X)$ is not an iso. But $\mathcal{O}(X) \simeq k[x,y]$ for if $f \in \mathcal{O}(X) \subset K(X) = k(x,y)$, $f = \frac{g}{h}$ irred (factoriality). $\Rightarrow$ for any other repres $f = \frac{g_1}{h_1} \Rightarrow h \mid h_1 \Rightarrow f$ not def at $Z(h) \Rightarrow h \in k$]. $\varphi : \mathbb{A}^1 \to X = Z(x^3 - y^2) \subset \mathbb{A}^2$, $\varphi(t) = (t^2, t^3)$ is bij but not iso!

(Prop) X var, $U \subset X$ open $\Rightarrow \dim X = \dim U$.

Pf. Put $n = \dim X$, let $\{P\} = Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n$. Let V aff nbhd of $P \Rightarrow \dim V = \dim X$. Let W affine open $\subset V \cap U \Rightarrow \dim U \geq \dim W = \text{tr. deg. } K(W)/k \overset{\text{Lect 4}}{=} \text{tr. deg. } K(V)/k = \dim V = \dim X$. Since by Lect 4 (Lect 4 & 2), $\dim U \leq \dim X$, qed.

(Def) A var X is called $\left\{\begin{array}{l}\text{separated}\\\text{complete}\end{array}\right.$ if $\left\{\begin{array}{l}\text{the diag of } X \times X \text{ is closed}\\ \forall \text{ var } Y \text{ the proj } X \times Y \to Y \text{ is closed } {}^{**}\end{array}\right.$

(Prop) A proj var is complete and separated. A quasi-proj var is sep.

Pf. Reduce to the case $X = \mathbb{P}^n$, $Y = \mathbb{A}^m$. Then use Lecture 5

(Prop) $f : X \to Y$ morph of var, X compl, Y sep $\Rightarrow f(X)$ is closed.

Pf. Then $\Gamma_f = \{(P,Q) \mid f(P) = Q\} \subset X \times Y$ is closed and apply ***

(Prop) Let $Y \subset \mathbb{P}^n$ be a proj var and $Y_i = Y \setminus Z(x_i)$. Then $\mathcal{O}(Y_i) = S(Y)_{(x_i)}$ and $\mathcal{O}(Y) = k$. Moreover $K(Y) = S(Y)_{(0)}$ ($= \Sigma^{-1} S(Y)$, $\Sigma = S(Y)^h \setminus \{0\}$).

Pf. Under the isom $k[y_1, \dots, y_n] \simeq k[x_0, \dots, x_n]_{(x_0)}$, $I(Y_0) \xrightarrow{\sim} I(Y)_{(x_0)} \Rightarrow A(Y_0) \simeq \frac{S_{(x_0)}}{I(Y)_{(x_0)}} \simeq S(Y)_{(x_0)}$

If $f \in \mathcal{O}(Y) \Rightarrow f$ morph : $Y \to \mathbb{A}^1$. Compose with $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. $\operatorname{Im}(Y \to \mathbb{P}^1)$ closed irred $\Rightarrow$ is a point.

(Cor) If $f : X \to Y$ morph X proj, Y affine $\Rightarrow f$ constant [Compose with $Y \subset \mathbb{A}^n \to \mathbb{A}^1$].

*** Take $X \times Y \xrightarrow{\varphi} Y \times Y$, $\varphi(x,y) = (f(x), y)$. Then $\Gamma_f = \varphi^{-1}(\Delta)$ diag in $Y \times Y$

** i.e. takes closed subsets into closed subsets

* If A domain $\Rightarrow \bigcap_{m \in \operatorname{Max} A} A_m = A$ for if $x \in Q(A) \setminus A \Rightarrow \underline{a} = \{b \in A \mid bx \in A\} \subsetneq A \Rightarrow \exists\, \underline{a} \subset m \subset A$ then $x \notin A_m$

(Ex) If $L_0, \dots, L_n \in S_1$ lin homog polyn & $V = Z(L_0, \dots, L_n)$ Then $f : \mathbb{P}^m \setminus V \to \mathbb{P}^n$, $f(P) = (L_0(P), \dots, L_n(P))$ is a morph of var (proof: with vertex ...)

Rem: Any proj is a comp of proj from pts!

Lecture 7. Complements about morphisms

(Def) A morph of var $f: X \to Y$ is called finite if $\forall P \in Y$ $\exists$ aff nbhd V s.t. $f^{-1}(V)$ affine & $\mathcal{O}(V) \to \mathcal{O}(f^{-1}(V))$ is finite.

(Cor) Noeth. normaliz in geom. form.

(Prop) Let $f: X \to Y$ finite & dominant (i.e. $f(X)$ dense in Y). Then f surj & has finite fibres. ($\dim X = \dim Y$)

Pf. May ass Y aff. Surj foll from Lect 2. (Concl = of dim) Finiteness of fibres foll from: Let $A \subset B$ finite, $m \subset A$ max. Suff B/mB has fin. many max ideals. But $A/m \to B/mB$ integral; $\Rightarrow \dim B/\mathfrak{q} = \dots$ $\Rightarrow$ $\mathrm{nil}(B/mB) = \bigcap$ all maximals $\overset{\text{Noeth}}{=} \bigcap$ fin many maximals $\Rightarrow$ q.e.d.

(Remark) A morph with fin. fibres need not be finite ($\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$).

(Def) If $f: X \to Y$ finite, can define its degree as $[K(X):K(Y)]$ ($<\infty$!)

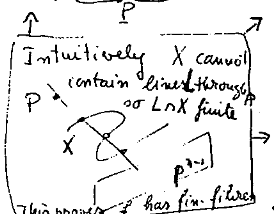
(Ex) $X \subset \mathbb{A}^2$, $Y = \mathbb{A}^1$, $X = Z(y^n - p(x))$, f = proj onto x-axis. Deg $f = n$. ¿p dist roots?

(α) (Prop) Let $f: X \to Y$ morph of var, f dominant. $\Rightarrow$ $f(X)$ contains an open set V s.t. $\forall y \in V$, $\dim f^{-1}(y) = \dim X - \dim Y$. [Not: $X_y = f^{-1}(y)$, closed!]

Pf. Suff X, Y affine. Put $A = \mathcal{O}(Y)$, $B = \mathcal{O}(X) = A[x_1, \dots, x_n, x_{n+1}, \dots, x_N]$ where $x_1, \dots, x_n$ alg ind/A and $x_{n+1}, \dots, x_N$ alg /$C = A[x_1, \dots, x_n]$. $\exists F \in C$, $F \neq 0$ s.t. $x_{n+1}, \dots, x_N$ integral /C_F. Let $g \in A$ be the product of all coeff of F. Then check $Y \setminus Z(g) \subset f(X)$ [if $m \subset A$ max $g \notin m \Rightarrow F \notin m[x_1, \dots, x_n] \Rightarrow m[x_1, \dots, x_n]_F = M \cap C_F$ where M max in B_F!]. Finally $\dim(B/mB)_F \overset{\text{int}}{=} \dim(C/mC)_F = n = \dim C - \dim A = \dim B - \dim A$

(Prop) Let $X \subset \mathbb{P}^n$ closed (= alg.), $E = Z(L_1, \dots, L_{n-d})$ be a d-dim lin. subspace $\subset \mathbb{P}^n$ s.t. $X \cap E = \emptyset$. Then the morph $f: X \hookrightarrow \mathbb{P}^n \setminus E \xrightarrow{\pi} \mathbb{P}^{n-d-1}$ is finite ($\pi(x) = (L_1(x), \dots, L_{n-d}(x))$ the proj).

Pf. May ass (induction) $d = 0$ (i.e. E a point). May assume $E = \{(\underbrace{1:0:\dots:0}_{P})\}$, $L_1 = x_1, \dots, L_n = x_n$ so $\pi(x_0 : x_1 : \dots : x_n) = (x_1 : \dots : x_n)$. $P \notin X \Rightarrow (x_1, \dots, x_n) \not\supset I(X)$



We want for all $i \geq 1$, $k[x_1, \dots, x_n]_{(x_i)} \to \left(\dfrac{k[x_0, \dots, x_n]}{I(X)}\right)_{(x_i)}$ finite. Suff to check $\dfrac{x_0}{x_i}$ (mod $I(X)$) integral over $\mathfrak{I}$. Pick $F \in I(X) \setminus (x_1, \dots, x_n)$ ($F \in R_d$) and write $F = F_0 + F_1 x_1 + \dots + F_n x_n + F_{11} x_1^2 + F_{12} x_1 x_2 + \dots$ where $F_0, F_i, F_{ij}, \dots \in (k[x_0])^h$, $\deg F_0 = d$, $\deg F_1 = \dots = \deg F_n = d-1$, a.s.o. So $F_0 = \lambda x_0^d$ ($\lambda \in k$ may ass $\lambda = 1$). $\Rightarrow \left(\frac{x_0}{x_i}\right)^d + \dots \equiv 0 \mod I(X)_{(x_i)}$ qed.

(Cor) (projective Noether normaliz) If X proj of dimension n $\Rightarrow$ $\exists$ finite surj morph $f: X \to \mathbb{P}^n$ (*)

Pf. $X \subset \mathbb{P}^N$. Choose $P \in \mathbb{P}^N \setminus X$, project X into $\mathbb{P}^{N-1}$, $\pi: X \xrightarrow[\text{fin}]{} \mathbb{P}^{N-1}$. By Lect 6 $\pi(X)$ closed (& repeat)

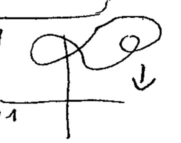
(Cor) If $X \subset \mathbb{P}^n$ closed of dimension d and $E \subset \mathbb{P}^n$ is a linear subv of $\dim E = n - d$ $\Rightarrow X \cap E \neq \emptyset$.

(Prop) Let $f: X \to Y$ morph of var, $\Gamma \subset X$ closed and ass. ($g: \Gamma \to Y$ the compos) $g^{-1}(P)$ are irred and of the same dimension r ($\forall P \in Y$) and that $g(\Gamma)$ is irred of dimension s. Then Γ is irred of dimension $r+s$ (closed if X, Y proj!)

Remark: Γ need not be irred Ex. Γ is not if $\dots$

(γ) For wor $\Gamma_c \to g(\Gamma)$, Γ_c = irr comp of Γ $\Rightarrow$ The Γ_c in which dim $< s+r$; those who dim have dim $s+r$, actually there is a unique comp dominating

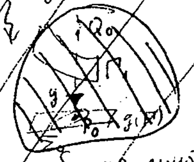
(*) In partic $\forall$ proj curve X $\exists$ finite surj mor $X \to \mathbb{P}^1$



We must prove $\dim B = r+s$. Make same constr. as in (α) $A \subset C \subset B$, $U \cap \Gamma \xrightarrow{g} V \to Y$, $C_F \subset B_F$

Take P corr to some max $m \subset A$, $m \not\ni g$.

$g^{-1}(P)$ irred $\Rightarrow r = \dim g^{-1}(P) = \dim g^{-1}(P) \cap U = n$



$g^{-1}(P) \cap U \neq \phi$ which happens for $P \notin g(U \cap \Gamma)$ $\supset$ open of $\overline{g(\Gamma)}$

$g^{-1}(P) \cap U$ corr to the ring B/mB so $\dim B/mB \overset{inv}{=} \dim (B/mB)_F \overset{ind}{=} \dim (C/mC)_F \overset{inv}{=} \dim \ldots = n$

So we must prove $\dim B = n + \dim A$. But $\dim B = \dim C = n + \dim A$ qed//. (use crit with tr.deg/k)

(Cor) X, Y var $\Rightarrow \dim X \times Y = \dim X + \dim Y$

(Cor) Let $X \subset \mathbb{P}^n$, $\dim X = r$. Then $\dim(\mathrm{Sec}(X)) \le 2r+1$ (Pf... see Lect 5 for $\mathrm{Sec}(X)$).

(Cor) Let $X \subset \mathbb{P}^n$ $\dim X = r$. Then $\exists$ lin sp. $E \subset \mathbb{P}^n$ of dim $n-2r-2$, $X \cap E = \phi$ & $\mathrm{proj}_E : X \to \mathbb{P}^{2r+1}$ from E is inj. [Pf...] (and cf. the above theory $\pi(X)$ is closed & $X \to \pi(X)$ is finite!) (later, more precise for X non-sing!). ↖ Analogue of "Whitney emb th" For curves $X \to \mathbb{P}^3$.

Rat. maps

(Rem) X, Y varieties (Y separ.), $\varphi, \psi : X \to Y$ morph. Ass $\varphi_{/U} = \psi_{/U}$ for $U \subset X$ open $\neq \phi$. Then $\varphi = \psi$.

Pf. $\{P \in X \mid \varphi(P) = \psi(P)\}$ is the preimage via $\varphi \times \psi : X \to Y \times Y$ of Δ_Y hence is closed $\Rightarrow$ qed.

(Def) X, Y var, Y separ. A rat map $X \dashrightarrow Y$ is an equiv class $\in \{(U, \varphi) \mid U \subset X, \varphi : U \to Y$ morph$\}$ where $(U, \varphi) \sim (V, \psi)$ if $\exists$ $W \subset U \cap V$ s.t. $\varphi_{/W} = \psi_{/W}$. By (Rem) above any 2 morph $X \to Y$ equal as rat maps coincide i.e. $\mathrm{Hom}(X,Y) \to \mathrm{Rat}(X,Y)$ is injective. (***)

A rat map is dominant if $\exists$ ($\Leftrightarrow$ all) repres. are so.

A birat map is .... 2 var are called bir equiv if ... A var is called rat... (*) unirat...

(Prop) There is equiv bet categ $\left\{ \begin{array}{l} \text{obj} = \text{sep. var} \\ \text{arrows} = \text{domin. rat. maps} \end{array} \right\}$ and $\left\{ \begin{array}{l} \text{obj} : \text{fields f.g.}/k \\ \text{arr} : \text{homo of such}/k \end{array} \right\}$ (By bir. iso ...)

Pf. Given var X assoc $K(X)$. Given $X \dashrightarrow Y$ repres by $U \to Y \Rightarrow K(Y) \to K(U)$. One can construct the inverse as follows. Take K/k f.g. $K = k(x_1, \ldots, x_n, y)$ where $x_1, \ldots, x_n$ alg ind y alg / $k(x_1, \ldots, x_n)$ [Th of prim elt] $\Rightarrow$ y root of $F \in k(x_1, \ldots, x_n)[t]$ F of smallest degree. Let $A = k[x_1, \ldots, x_n, t]$ and $X = Z(F) \subset \mathbb{A}^{n+1}$. We actually got: (**)

(Prop) Any sep. var Y (of dim n) is birationally iso to a hypersurface in $\mathbb{A}^{n+1}$ (or $\mathbb{P}^{n+1}$ if we want).

(Def) of curves & non-sing curves

(Prop) X nonsing curve, Y proj var. Then any rat map $X \dashrightarrow Y$ is everywhere defined. (4)

Pf. We may ass $Y = \mathbb{P}^n$. Ass $X \dashrightarrow \mathbb{P}^n$ is repres by $\varphi : U \to \mathbb{P}^n$. So $X \setminus U$ finite. Let $P \in X \setminus U$ Choose j s.t. $\varphi(U) \cap H_j \neq \phi$ $H_j = Z(x_j) \subset \mathbb{A}^n$ (Ass $j = 0$) Then φ induces a mor $U_0 \to \mathbb{P}^n \setminus H_0 = \mathbb{A}^n$ hence is given by reg fns $\varphi_1, \ldots, \varphi_n$ on $U_0 = \varphi^{-1}(\mathbb{P}^n \setminus H_0)$. Then $(U, \varphi) \sim (W, \psi)$ where $\psi(Q) = (\psi_0(Q) : \cdots : \psi_n(Q))$ and W = open set where ψ_i are reg and do not vanish simultaneously ($P \in W$!)

So $\varphi(Q) = (1 : \varphi_1(Q) : \cdots : \varphi_n(Q))$ for $Q \in U_0$. Now $\varphi_i \in K(X)$ so we may write $\varphi_i = \psi_i / \psi_0$ where $\psi_i, \psi_0 \in \mathcal{O}_P$ and s.t. not all $\psi_0, \ldots, \psi_n$ are divisible by the parameter of $\mathcal{O}_P$.

(4) This fails for $\dim X \ge 2$!

(***) A rat map $\in \mathrm{Rat}(X,Y)$ is called everywhere defined (or regular) if it is in the image of $\mathrm{Hom}(X,Y)$. This is equiv to it being repres by $\{(U_i, \varphi_i)\}$ where U_i is an open cov of X (by Remark above).

(**) hypers. means Z(a single pol).

(*) We shall prove later for curves unirat $\Rightarrow$ rat. It is true for dim = 2 (surf). It fails for dim ≥ 3 (Griff. Clem., Manin, Isk Mumford, Artin).

Lecture 8. Non-singular projective models of [illegible]

(Def) Let K/k be a f.g. field ext. (tr.deg $K/k = 1$) By a non-sing proj model of it we understand a non-s. proj curve X s.t. $K(X) \cong_k K$. By a non-sing proj model of a curve X we und a n.p.m of $K(X)$.

(Cor) (of Lect 7) If X_1, X_2 are nonsing proj mod of K, tr.deg $K/k = 1 \Rightarrow X_1 \cong X_2$ as var.

(Th) Any K with tr.deg $K/k = 1$ has a nonsing proj model (unique by ↑) (*). In other words any curve Y is birationally equiv to a non-sing proj curve X and X is unique up to iso of var!

We prove more (α)

(Def) A var X is normal iff $\mathcal{O}_{X,P}$ is integrally closed $\forall P \in X$. By a normalis we mean a finite mor $X^{nor} \to X$, X^{nor} normal & $\forall$ other $Y \to X$ $\exists!$ $Y \to X^{nor}$ making $\underset{nor}{\ldots}$

Pf see (β) verso → (Prop) Any affine var has a norm which is affine [illegible]

(α) (Th) Any proj —"— —"— —"— projective.

We need lemmas. Recall def of $S_{\geq 0}$, $S_+ = S_{>0}$, $S[d]$ for S a grad. alg ($S[d] = \bigoplus S_{di}$)

(L1) Let $S = \bigoplus_{i \geq 0} S_i$ graded f.g. $/k$, $S_0 = k$. Then there $\exists$ $d \in \mathbb{N}$ s.t. $S[d]$ [illegible] is gen over k by $S[d]_1 = S_d$.

Pf. Ass $x_1, \ldots, x_n$ gener of deg $d_1, \ldots, d_n$, $d' = \text{smallest common multiple}(d_1, \ldots, d_n)$, $d = nd'$. (This d will do. Indeed) $\forall$ elt in S_N is a sum of monomials $x_1^{f_1} \cdots x_n^{f_n}$ with $\sum_{i=1}^n f_i d_i = N$ trivially. Let $e_i = \dfrac{d'}{d_i}$. If all $f_i < e_i \Rightarrow N < d$. Hence if $N \geq d \Rightarrow \underbrace{\prod x_i^{f_i}}_{\in S_N} = \underbrace{x_{i_1}^{e_{i_1}}}_{\in S_{d'}} \underbrace{\prod x_i^{f_i'}}_{\in S_{N-d'}}$. Applying n times (and assuming $N \geq 2d$) this $\Rightarrow$

$$\underbrace{\prod x_i^{f_i}}_{\in S_N} = \underbrace{x_{i_1}^{e_{i_1}} x_{i_2}^{e_{i_2}} \cdots x_{i_n}^{e_{i_n}}}_{\in S_d} \cdot \underbrace{(\text{something})}_{\in S_{N-d}}.$$

Hence if $N =$ multiple of d, $\prod x_i^{f_i}$ is a product of monomials in S_d.

(L2) Let S be as in (L1) and $f \in S[d]_1 = S_d$. Then $S_{(f)} = S[d]_{(f)}$.

Pf. Clearly "$\supset$". Conversely if $\frac{a}{f^n} \in S_{(f)} \Rightarrow \deg a = n \cdot d \Rightarrow a \in S[d]$

(L3) Let $R = \bigoplus_{d \geq 0} R_d$ be graded Noeth. domain, $\Sigma \subset R^h$ a mult. closed subset and S the integral clos of R in the (graded) ring $(\Sigma^{-1}R)_{\geq 0}$ (= pos part of $\Sigma^{-1}R$). Then S is a graded subring of $(\Sigma^{-1}R)_{\geq 0}$

Pf. Let $x \in S$, $x = x_0 + \cdots + x_m$, $x_m \neq 0$, x_i hom comp. We want $x_i \in S$. By Lect 3, $R[x]$ is a f.g. R-module generated say by $r_1/s_1, \ldots, r_k/s_k$ where $r_i \in R$, $s_i \in \Sigma$. We get with [illegible] that $sx^n \in R$ for all $n \geq 1$. Write $s = s_0 + \cdots + s_p$, $s_p \neq 0$, as sum of hom comp.

$\Rightarrow (s_0 + \cdots + s_p)(x_0 + \cdots + x_m)^n \in R$. Taking hom comp of deg $p + nm \Rightarrow s_p x_m^n \in R$ for all n

$\Rightarrow x_m^n \in \frac{1}{s_p} R \Rightarrow R[x_m] \subset R + \frac{1}{s_p} R \Rightarrow R[x_m]$ f.g. R-mod $\Rightarrow x_m$ integral $/R$

$\Rightarrow x_m \in S$. By induction all $x_i \in S$.

(*) $\exists$ holds for tr.deg K/k arbitr. (Zar., Hironaka) with a suitable notion of non-sing.

(L4) Let $R \subset S$ graded domains, S finite over R, both f.g. $/k$. Let $f_1, \dots, f_k \in R$ s.t. $R_+ = \sqrt{(f_1, \dots, f_k)}$. Then $S_+ = \sqrt{(f_1,\dots,f_k)S}$. Moreover $R_{(f_i)} \subset S_{(f_i)}$ are integral.

Pf. Let $x \in S_d \Rightarrow x^n + a_1 x^{n-1} + \dots + a_n = 0$, $a_i \in R$. Taking degree $nd \Rightarrow$ we may ass a_i homogeneous $\Rightarrow a_i \in R_+$ so $x^n \in R_+ S$ so $x \in \sqrt{R_+ S} = \sqrt{(f_1,\dots,f_k)S}$.
Now $R \subset S$ finite $\Rightarrow R_{f_i} \subset S_{f_i}$ finite and conclude by ;

(L5) $R \subset S$ graded (non neces. positive) domains S integral $/R \Rightarrow S_0$ integral $/R_0$

Pf. $R_0 \subset S_0$ integral bec if $x \in S_0$, $x^n + a_1 x^{n-1} + \dots + a_n = 0$, $a_i \in R$. Taking $\deg = 0 \Rightarrow$ may ass $a_i \in R_0$.

(L6) $R \subset S$ graded (non nec pos.) domains R integrally closed in $S \Rightarrow \begin{cases} R_{\geq 0} \text{ int cl in } S_{\geq 0} \\ R_0 \text{ int cl in } S_0 \end{cases}$

Pf of Th (2). Let $X \subset \mathbb{P}^n$ be a proj var. Let $\Sigma = S(X)^h \setminus \{0\}$ and S' = int. clos of $S(X)$ in $(\Sigma^{-1} S(X))_{\geq 0}$. By (L3) S' is graded. By (3) below S' is finite over $S(X)$ [because $S' \subset$ int. clos of $S(X)$]. Let d be s.t. (L1) & (L2) hold i.e. $S'[d]$ gen by $S'[d]_1 = S'_d$ ($\Rightarrow S'[d]_{(f)} = S'_{(f)}$ for $\forall f \in S'_d$). Let $S'[d] = k[x_0, \dots, x_N]/\mathfrak{a}$ for $\mathfrak{a}$ prime and $Y = Z(\mathfrak{a}) \subset \mathbb{P}^N$. We claim Y is a normalis of X. Let $f_1, \dots, f_k \in S(X)_d$ be a k-basis (so $\sqrt{(f_1,\dots,f_k)} = S(X)_+$).
By (L4) $S'_+ = \sqrt{(f_1,\dots,f_k)S'}$ so $\Rightarrow S'[d]_+ = \sqrt{(f_1,\dots,f_k)S'[d]}$ $Y \setminus Z(f_1), \dots, Y \setminus Z(f_k)$ is an affine covering of Y. We have $\mathcal{O}(Y \setminus Z(f_i)) = S(Y)_{(f_i)} = S'[d]_{(f_i)} = S'_{(f_i)}$

By (L4) $S'_{(f_i)}$ is integral over $S(X)_{(f_i)}$. It is suff to check that the last 2 rings have same qt field & the 1st is normal. Now $Q(S'_{(f_i)}) = (\Omega^{-1} S')_0$ where $\Omega = (S')^h \setminus \{0\}$ and $Q(S(X)_{(f_i)}) = (\Sigma^{-1} S(X))_0$. Clearly $(\Sigma^{-1} S(X))_0 \subset (\Omega^{-1} S')_0$. Conversely, if $\frac{s'}{w} \in (\Omega^{-1}S')_0$, $w, s' \in S'_e \Rightarrow w, s' \in (\Sigma^{-1} S(X))_{\geq 0} \Rightarrow$
$w = \frac{s_1}{\sigma_1}$, $s' = \frac{s_2}{\sigma_2} \Rightarrow s'/w = \frac{s_1 \sigma_2}{s_2 \sigma_1} \in (\Sigma^{-1}S(X))_0$. To check $S'_{(f_i)}$ normal, i.e. int cl in $(\Sigma^{-1}S(X))_0$, it is suff to check by (L6) that S'_{f_i} integr cl in $\Sigma^{-1} S(X)_{\geq 0}$ which is clear (fractions of int cl are int cl)

f.i $\in S(X) \subset S(X)_{(f_i)}$; $f_i \in S'[d]$; $k[x]/\mathfrak{a}$, $Y = Z(\mathfrak{a})$

(3) (Def) $K \subset F$ fin ext, $x \in F$; $Tr_{F/K}(x) = Tr(\phi_x : F \to F)$, $\phi_x(y) = xy$. (Th) $K \subset F$ fin sep $\Rightarrow (x,y) \mapsto Tr(xy)$ non-deg. (Pf. Any book on fields)
(P) In char 0 $Tr(x) = [F : K(x)] \cdot \sum x_i$ ($x_1 = x, x_2, \dots, x_n$ conj of x in $\overline{K} \supset F$). (Pf: any book on fields)
(Rem) If –"– $K = Q(A)$, A int closed, x int $/A \Rightarrow Tr(x) \in A$ [Pf: $x_i = \sigma_i(x)$ where $\{\sigma_1, \dots, \sigma_n\} = Gal(K(x_1, \dots, x_n)/K)$ so x_i int $/A$ so $Tr(x)$ int $/A \Rightarrow \in A$]
(Th) (char 0) A Noeth int cl, $K = Q(A)$, F/K fin, A' = int cl of A in F. Then A' is fin $/A$.
Pf. Let $u_1, \dots, u_n \in A'$ basis F/K & $v_1, \dots, v_n$ dual basis wrt $(x,y) \mapsto Tr(xy)$ so $Tr(u_i v_j) = \delta_{ij}$
Claim $A' \subset \sum A v_j$ (and this $\Rightarrow$ Th). Let $x \in A'$ so $x = \sum x_j v_j$, $x_j \in K$ $\Rightarrow$ $x u_i \in A'$ $\overset{(Rem)}{\Rightarrow} Tr(xu_i) \in A$. But $Tr(x u_i) = Tr(\sum x_j v_j u_i) = x_i$. So $x \in \sum A v_j$.
(Th) (char 0) A = a f.g. k-alg integral, $F = Q(A)$, A' = int cl of A $\Rightarrow A'$ is fin $/A$
Pf Noether $k[x_1, \dots, x_n] \subset A$ fin. Put $K = k(x_1, \dots, x_n)$. Then A' = int clos of $k[x_1, \dots, x_n]$ in F so it is fin $/ k[x_1, \dots, x_n]$ hence $/A$.

Lecture 9. Divisors on curves. Bezout Theorem.

(bec. it contains an open set and has dim 1)

~~[crossed-out line]~~

Prop Any non-constant morph $f: X \to Y$ of nonsing proj curves is finite.

Pf. $f(X) = Y$ by Lect. 6. Let $V \subset Y$ be affine[3], $A = \mathcal{O}(Y)$, $B =$ integr clos of A in $Q(X)$ and U be affine with $\mathcal{O}(U) = B$. Since $K(U) = K(X)$, $\exists$ birat map $\varphi: U \dashrightarrow X$. Since U non-sing & X proj $\Rightarrow$ φ morph & $\varphi(U)$ open in X. ~~[crossed-out line]~~ It is suff to check $\varphi(U) = f^{-1}(V)$. Clearly "$\subset$". Ass "$\not\supset$" so pick $Q_0 \in f^{-1}(V) \setminus \varphi(U)$

$$\begin{array}{c} X \xleftarrow{\varphi} U \\ \downarrow \quad\ \ \downarrow \\ Y \supset V \end{array}$$

$P_0 = f(Q_0) \in V$, $f^{-1}(P_0) = \{Q_0, \underbrace{Q_1, \dots, Q_r}_{\not\subset U}, \underbrace{Q_{r+1}, \dots, Q_n}_{\subset U}\}$. Take $w \in K(X)$ s.t. $w \notin \mathcal{O}_{Q_0}$ but $w \in \mathcal{O}_{Q_{r+1}} \cap \dots \cap \mathcal{O}_{Q_n}$ (*). Let $R_1, \dots, R_k \in U \simeq \varphi(U)$ be the pts where $w: U \dashrightarrow k$ is not defined (so $f(R_i) \neq P_0, \forall i$). Then we can find $v \in \mathcal{O}(V)$ s.t. $v(P_0) \neq 0$ (i.e. $v \in \mathcal{O}^*_{P_0}$) and $vw \in \mathcal{O}(U) \overset{f.t.}{=} B$.

R_i, Q_{r+2}, Q_{r+1}, $U \simeq \varphi(U)$, Q_0, V, P_0

So $u = vw$ is integral $/A$ $\Rightarrow$ $u^n + a_1 u^{n-1} + \dots + a_n = 0$, $a_i \in A$ $\Rightarrow$ $u = -a_1 - a_2/u - \dots - a_n/u^{n-1}$. But $u \notin \mathcal{O}_{Q_0}$ while $-a_1 - a_2/u - \dots - a_n/u^{n-1} \in \mathcal{O}_{Q_0}$ (because $u^{-1} \in \mathcal{O}_{Q_0}$) $\not\!\!X$.

Def X a non-sing proj curve. By a div on X, an elt of the free ab gr gen by X i.e. $D = \sum_{P \in X} n_P P$, $n_P \in \mathbb{Z}$ (all but fin many are 0). Put $\deg D = \sum n_P$ (Supp D = ...). Put $\mathrm{Div}(X) =$ gr of div. Def also $D_1 + D_2$ and $D \geq 0$.

(write $n_P = v_P(D)$)

Def Ass in $\uparrow$ $X \subset \mathbb{P}^n$ & ass $H \subset \mathbb{P}^n$ is a hypers of degree N, $H \not\supset X$. Define a div $X \cdot H \in \mathrm{Div}(X)$ as foll. If $P \in X \cap H$ define $(X \cdot H)_P = v_P(h)$ where v_P is the val of $\mathcal{O}_P$ & if $H = Z(F)$, $F \in S_N$ then h is the image via $\mathcal{O}_{\mathbb{P}^n, P} \to \mathcal{O}_{X,P}$ of $\frac{F}{x_i^N}$ where i is any index s.t. $P \notin Z(x_i)$. Then put $X \cdot H = \sum_{P \in X \cap H} (X \cdot H)_P P$; it is called the hypers. section; clearly $\mathrm{Supp}\, X \cdot H = X \cap H$. (Ex): hyperpl sect of plane curves of degree d.

Def Let $f: X \to Y$ be (non-ct) mor of non-sing proj curves. We define a map $f^*: \mathrm{Div}(Y) \to \mathrm{Div}(X)$ as follows. If $D = Q \in Y$ and $t_Q \in \mathcal{O}_Q$ parameter then put $f^* Q = \sum_{f(P) = Q} v_P(f^* t_Q) P$ where v_P val of $\mathcal{O}_P$ (note $\mathcal{O}_Q \overset{f^*}{\subset} \mathcal{O}_P$). & extend by lin.

$$\begin{array}{c} X \\ \downarrow \\ Y \end{array}$$

Def X non-sing proj curve. Define a gr. homo $K(X)^* \xrightarrow{\alpha} \mathrm{Div}(X)$, $g \mapsto (g) := \sum_{P \in X} v_P(g) \cdot P$[4]. We have $\operatorname{Ker} \alpha = k^*$ (bec $\mathcal{O}(X) = k$). Define $P(X) = \operatorname{Im} \alpha$; call them princ div. Define $Cl(X) = \mathrm{Div}(X)/P(X)$ the div. class gr. Two div $D_1, D_2 \in \mathrm{Div}(X)$ are called lin eq if $D_1 - D_2 \in P(X)$. Write $D_1 \sim D_2$.

(set of poles and zeroes, ords)

Prop Let $f: X \to Y$ be mor of (non-ct) non-sing proj curves. For $g \in K(Y)$ we have $f^*((g)) = (f^* g)$ so $\mathrm{Div}(Y) \to \mathrm{Div}(X)$ factors through $Cl(Y) \to Cl(X)$.

(4) Put $(f) = (f)_+ - (f)_\infty$ ($(f)_-$) "zeroes and poles" of f on X. Ex $X = \mathbb{P}^1$, $f = \frac{x_1}{x_0} \Rightarrow (f) = \underset{(1:0)}{0} - \underset{(0:1)}{\infty}$

(3) $V = Y \setminus$ hyperplane H

(**) View $Y \subset \mathbb{P}^n$ and take hyperplanes $H_i \ni f(R_i)$ not passing through P_0. ~~and another hyp~~ Then put $\tilde{v} = \frac{H_1^N H_2^N \cdots H_m^N}{H^{mN}}$ for $N \gg 0$ and restrict it to a v in V.

(*) one can take w ~~[crossed out]~~ because: (L) If $X \overset{\subset\mathbb{P}^n}{}$ non-sing proj curve & $Q_0, Q_1, \dots, Q_n \in X$ dist pts $\Rightarrow \exists\, w \in K(X)$ s.t. $w \notin \mathcal{O}_{Q_0}$, $w \in \mathcal{O}_{Q_1} \cap \dots \cap \mathcal{O}_{Q_n}$ [Pf: Choose ~~[crossed out]~~ hyperplanes $H_0, H_1, \dots, H_n \subset \mathbb{P}^n$, $H_i \ni Q_i$ & ($H_i \not\ni Q_j$ for $j \neq i$) & let $\tilde{w} = \frac{H_1 \cdots H_n}{H_0^n} \in \mathcal{O}(\mathbb{P}^n \setminus Z(H_0))$. Then restrict $\tilde{w}$ to X.

Pf. $f^*((g)) = f^*(\sum_{Q\in Y} v_Q(g)Q) = \sum_{Q\in Y} v_Q(g) f^*Q = \sum_{\substack{f(P)=Q\\ Q\in Y}} v_Q(g) v_P(t_Q) P = \sum_{P\in X} v_P(f^*(g)) P = (f^*(g))$.

Prop. If $X \subset \mathbb{P}^n$ non-sing and $H_1, H_2 \subset \mathbb{P}^n$ hypers of the same degree $\overset{(X \not\subset H_i)}{\Rightarrow}$ $X\cdot H_1 \sim X\cdot H_2$

Pf. $H_1/H_2 \in K(\mathbb{P}^n)$ induces a rat fun g on X. By def $X\cdot H_1 - X\cdot H_2 = (g)$ qed.

Picture

Prop. Let $f: X \to Y$ non-ct mor of non-sing proj. curves. For all $D \in \mathrm{Div}(Y)$ we have $\deg f^*D = (\deg f)(\deg D)$. (recall $\deg f := [K(X):K(Y)]$.)

Pf. Suff $D = Q \in Y$ so suff $\deg f^*Q = \deg f$; f affine $\Rightarrow \exists\, V \subset Y$ aff nbhd of Q s.t. $U = f^{-1}(V)$ affine. Put $A = \mathcal{O}(V)$, $B = \mathcal{O}(U)$. Let $m = m_Q \in \mathrm{Max}(A)$. Since B = int clos of A in $K(X)$ by Lect 2, B_m = int clos of A_m and f.g. as A_m-module $\Rightarrow$ B_m free A_m-mod (*) of rk $= r = [K(X):K(Y)]$ (**) $\Rightarrow$ B_m/mB_m is an A/m-vect space of dim. r (***). The primes ($\Leftrightarrow$ max) of B_m/mB_m are in bij with the pr. $\{m_1, \dots, m_k\}$ of B lying over m (Lect 1) hence with $\{P \in X \mid f(P) = Q\}$. Let $(t) = mA_m$. Now $\dfrac{B_m}{mB_m} = \dfrac{B_m}{f^*t\, B_m} = \prod \dfrac{B_{m_i}}{f^*t\, B_{m_i}} = \prod \dfrac{\mathcal{O}_P}{f^*t\,\mathcal{O}_P}$. Taking $\dim_k$ $\Rightarrow$ $r = \sum \dim \dfrac{\mathcal{O}_P}{f^*t\,\mathcal{O}_P} \overset{\text{Lect 3}}{=} \sum v_P(f^*t) = \deg f^*Q$.

(L) Let C be a Noeth ring of dim 0 and $m_1, \dots, m_k$ its primes. Then $C \xrightarrow{\alpha} \prod C_{m_i}$ is iso.

Pf. By Lect 4 $\Rightarrow \sqrt{0} = m_1 \cap \dots \cap m_k = m_1 \cdots m_k$; Noeth $\Rightarrow \exists N$ s.t. $m_1^N \cdots m_k^N = 0$. Note $m_i + m_j = C$ $(i \neq j)$ $\Rightarrow \forall N$ $m_i^N + m_j^N = C$. Consider the morph $\alpha: C \to \prod C/m_i^N$. If we prove α iso we are done [because C/m_i^N are local. Easy $\Rightarrow C_{m_i} \simeq C/m_i^N$ for $N \gg 0$]. To check α inj we must check $\bigcap_{i=1}^k m_i^N = 0$. We check it for $k=2$ (for $k \geq 3$ induction (*)). Since $m_1^N + m_2^N = (1) \Rightarrow \exists\, x + y = 1$, $x \in m_1^N$, $y \in m_2^N$. Let $z \in m_1^N \cap m_2^N \Rightarrow z = xz + yz \in m_1^N \cdot m_2^N = 0$. To check surj of α let' find $x \in C$ s.t. $\alpha(x) = (1, 0, \dots, 0)$. $\exists\, u_1 + v_i = 1$ $(i > 1$, $u_1 \in m_1^N$, $v_i \in m_i^N)$. Take $x = v_2 v_3 \cdots v_k$. Then $x = (1-u_1)(1-u_1)\cdots(1-u_1) \equiv 1 \bmod m_1^N$ so $\alpha(x) = (1, 0, \dots, 0)$.

Prop. X non-sing proj curve & $D_1, D_2 \in \mathrm{Div}(X)$, $D_1 \sim D_2 \Rightarrow \deg D_1 = \deg D_2$

Pf. $D_1 - D_2 = (g)$. Think $g: X \dashrightarrow \mathbb{A}^1$. X nons $\Rightarrow g: X \to \mathbb{P}^1$. We have $(g) = g^*(0 - \infty)$ so $\deg(g) = (\deg g)(\underbrace{\deg(0-\infty)}_{0}) = 0$. Cor Princ div have degree 0!

Cor. $X \subset \mathbb{P}^n$ non-sing proj curve, $H_1, H_2 \subset \mathbb{P}^n$ hypers of same degree $\overset{(H_i \not\supset X)}{\Rightarrow}$ $\deg X\cdot H_1 = \deg X\cdot H_2$

Def. $X \subset \mathbb{P}^n$ — " — — " —. Define $\deg X := \deg X\cdot H$ (where H any hyperplane $\not\supset X$).

Ex. If $n = 2$, $X = Z(g) \subset \mathbb{P}^2$, $\deg g = d$. Then $\deg X = d$.

Th. (Bézout) $X \subset \mathbb{P}^n$ non-sing proj of degree d_1 & $H \subset \mathbb{P}^n$ hypers of deg d_2, $X \not\subset H \Rightarrow \deg X\cdot H = d_1 d_2$ in partic $\#(X \cap H) \leq d_1 d_2$ and $X \cap H \neq \emptyset$!

(*) What must be proved is that if A ring, $\underline{a}_1, \dots, \underline{a}_k \subset A$ ideals s.t. $\underline{a}_i + \underline{a}_j = A$ $(i \neq j)$ $\Rightarrow \prod \underline{a}_i = \bigcap \underline{a}_i$. We proved in the text that $A \to \prod (A/\underline{a}_i)$ is surj! This is the Chinese remaind th.

(***) if M free A-mod of rk r & $m \subset A$ max $\Rightarrow$ M/mM vect sp of dim r / A/m.

(**) S^{-1} commutes with $\oplus$

(*) A f.g. A-mod M is called free if $\simeq A^n$ (n = rank); it is called torsion free if $\forall a \neq 0 \in A$, $\forall x \neq 0 \in M \Rightarrow ax \neq 0$. A is called princ if $\forall$ ideal is princ i.e (a).
Th A princ, M f.g. torsion free $\Rightarrow$ M free (see *Algebra: Gr, Rings, Fields)

Lecture 10. Curves in projective plane

(Def) $F \in k[x_0,x_1,x_2]$ hom of deg d, $F \neq 0$ irred, $X = Z(F) \subset \mathbb{P}^2$. Call it line, conic, cubic, ... (Call X (plane proj) curve.)

"Recall" X called non-sing at $P \in X$ if $\mathcal{O}_{X,P}$ DVR. Call $\mathrm{Sing}(X) = \{P \mid X$ is sing at $P\}$.

(Lemma) (Euler ident.) $x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} = dF$ [Pf. suff for $F = x_0^{i_0}x_1^{i_1}x_2^{i_2}$].

(α) (Prop) $\mathrm{Sing}(X) = Z(\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2})$.

Pf. Want $P \in \mathrm{Sing}(X)$ iff $\frac{\partial F}{\partial x_i}(P) = 0\ \forall i$. May ass $P = (1:0:0)$. Put $y_1 = \frac{x_1}{x_0}$, $y_2 = \frac{x_2}{x_0}$, $f(y_1,y_2) = F(1,y_1,y_2)$
$\Rightarrow \frac{\partial F}{\partial x_i}(1,y_1,y_2) = \frac{\partial f}{\partial y_i}(y_1,y_2)$ (check for $F = x_0^{i_0}x_1^{i_1}x_2^{i_2}$!). Lect 6 $\Rightarrow$ $P \in \mathrm{Sing}(X) \Leftrightarrow f(0,0) = \frac{\partial f}{\partial y_1}(0,0) = \frac{\partial f}{\partial y_2}(0,0) = 0 \Leftrightarrow$
$\Leftrightarrow F(1,0,0) = \frac{\partial F}{\partial x_1}(1,0,0) = \frac{\partial F}{\partial x_2}(1,0,0) = 0 \Leftrightarrow$ by Euler.

(Def) By a proj transf of $\mathbb{P}^n$ we mean .... Let $PGL(n+1) = \ldots$. Two curves called projectively equiv ...
(Ex) $\mathrm{Aut}\,\mathbb{P}^1 = PGL(2)$

E (Prop) Any line in $\mathbb{P}^2$ is proj equiv to $Z(x_0)$. Any conic (irr!) is pr equiv to $Z(x_0^2+x_1^2+x_2^2)$ [Pf]

(Prop) $\forall$ line or conic is $\simeq \mathbb{P}^1$ [for conic $Z(x_0^2+x_1x_2) \dashrightarrow \mathbb{P}^1$, $(x_0:x_1:x_2) \mapsto (x_0:x_1)$ proj from $(0:0:1)$]

(Def) $X \subset \mathbb{P}^2$ curve, $P \in X$, $P \notin \mathrm{Sing}(X)$. Call "proj tg line" at P the line $\overline{T_P}X := Z(\frac{\partial F}{\partial x_0}(P)x_0 + \frac{\partial F}{\partial x_1}(P)x_1 + \frac{\partial F}{\partial x_2}(P)x_2)$
Euler $\Rightarrow$ $P \in \overline{T_P}X$ [Ex ...]. A line $L \subset \mathbb{P}^2$ called tg to X at P if $L = \overline{T_P}X$.
Called tg to X if $\exists P$ s.t. $L = \overline{T_P}X$.

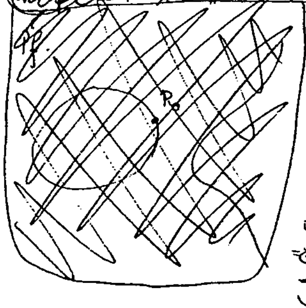
(δ) (Prop) $X \subset \mathbb{P}^2$ curve, $P \in X \setminus \mathrm{Sing}(X)$, $L \subset \mathbb{P}^2$ line. L tg to X at P iff $(X \cdot L)_P \geq 2$.

Pf. May ass $P = (1:0:0)$; if F, f as in (α) and $U = \mathbb{P}^2 \setminus Z(x_0) \Rightarrow \overline{T_P}X \cap U = Z(\underbrace{\frac{\partial f}{\partial y_1}(0,0)}_{a}y_1 + \underbrace{\frac{\partial f}{\partial y_2}(0,0)}_{b}y_2)$
Lect 3 (if $m = \mathcal{O}_{X,P}$) $\Rightarrow m/m^2 = \frac{ky_1+ky_2}{k(ay_1+by_2)}$. Ass $L = Z(\alpha_0x_0+\alpha_1x_1+\alpha_2x_2) \Rightarrow L \cap U = Z(\alpha_0+\alpha_1y_1+\alpha_2y_2)$.
So $(X\cdot L) \geq 2 \Leftrightarrow \alpha_0 = 0$ & $\alpha_1y_1+\alpha_2y_2|_X \in m^2 \Leftrightarrow \alpha_0 = 0$ & $\exists \lambda, \alpha_1 = \lambda a, \alpha_2 = \lambda b \Leftrightarrow$ q.e.d.

(Def) X non-sing proj curve. We have $\deg: \mathrm{Div}(X) \twoheadrightarrow \mathbb{Z}$. It induces $\deg: Cl(X) \to \mathbb{Z}$ (Lect 9)

(β) (Prop) —"— —"— The foll are equiv: 1) $X \simeq \mathbb{P}^1$, 2) X bir isom $\mathbb{P}^1$ (by def rational), 3) $Cl^0(X) = 0$, 4) $\forall P,Q \in X$, $P \sim Q$, 5) $\exists P,Q \in X$, $P \sim Q$, $P \neq Q$! . In partic this holds for lines & conics.

Pf. 1) $\Leftrightarrow$ 2) by Lect 7, 3) $\Leftrightarrow$ 4) obv., 1) $\Rightarrow$ 4) if $P = (a:b)$, $Q = (c:d)$ let $f = \frac{ax_0 - bx_1}{cx_0 - dx_1} \Rightarrow (f) = P - Q$
4) $\Rightarrow$ 5) obv ; 5) $\Rightarrow$ 1) Let $(f) = P - Q$. f defines $f: X \to \mathbb{P}^1$ ~~...~~. Suff $\deg f = 1$ (bec f finite (Lect 9) & $\mathbb{P}^1$ normal!). But $1 = \deg f^*\mathcal{O} = (\deg f)\deg \mathcal{O} \Rightarrow$ q.e.d.//.

(γ) 

(Prop) The cubic $X = Z(x_2^2x_0 - x_1^3 + x_1x_0^2) \subset \mathbb{P}^2$ is non-sing and not rational. [later a more conceptual proof!]

Pf. Non-sing clear. For non-rat suff to check $A = k[x,y]/(y^2-x^3+x) = k[\bar{x},\bar{y}]$ is not factorial [bec $\mathcal{O}(U)$, $U \subset \mathbb{P}^1$ is fact!]. We have $k[x] \hookrightarrow A$, $\sigma: A \to A$ $\sigma(\bar{x}) = \bar{x}$, $\sigma(\bar{y}) = -\bar{y}$. Put $N: A \to k[x]$, $N(u) = u\cdot\sigma(u)$ ("norm"; it is multiplic). We have A^* (= units of A) $= k^*$ [$uv=1 \Rightarrow 1 = N(1) = N(uv) = N(u)N(v) \Rightarrow$?]

(2) So if $u = g(\bar{x}) + \bar{y}h(\bar{x}) \Rightarrow N(u) = [g(\bar{x})+\bar{y}h(\bar{x})][g(\bar{x})-\bar{y}h(\bar{x})] = g(x)^2 - (x^3-x)h(x)^2$

(1) Any quadr form — either has a square $x_0^2 \Rightarrow q = x_0^2 + 2x_0L + q_1$ put $x_0' = x_0 + L \Rightarrow q = (x_0')^2 + q_1 - L^2$ a.s.o.
— has no square $\Rightarrow q = x_0x_1 + \ldots$. Put $x_0 = x_0' + x_1'$, $x_1 = x_0' - x_1'$ ..

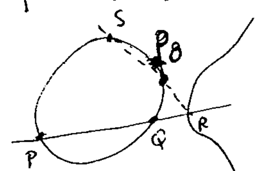
$N(u) = \lambda \in k^* \Rightarrow g(x)$ ...

Now x inv in A [if $x = \underset{\text{non-inv}}{uv} \Rightarrow x^2 = N(x) = N(u)N(v) \Rightarrow N(u) = \underset{\neq 0}{\lambda} x \Rightarrow g^2 = (x^3 - x)h^2 + \lambda x \Rightarrow$ as above]
$u = g + yh$

& x not prime in A [$A/(x) = k[y]/(y^2)$!]. (1)

(Prop) Let X be the cubic above, $P_0 \in X$. Then $X \to Cl^0(X)$, $P \mapsto Cl(P - P_0)$ is bij.

Pf. Inj by (β) & (γ). Surj. Step 1. Given $P, Q \in X$, $\exists S \in X$ s.t. $P + Q \sim P_0 + S$



(*)

See ($L = \overline{PQ}$ or $\overline{T_P X}$ if $P = Q$, write $X \cdot L = P + Q + R$ (φ), let $L_1 = \overline{P_0 R}$ or $\overline{T_R X}$ if $P_0 = R$, — " — $X \cdot L_1 = P_0 + R + S \Rightarrow P + Q + R \sim P_0 + R + S \Rightarrow$ qed).

Step 2 $\forall D \in Div(X)$, $D \geq 0 \Rightarrow D \sim kP_0 + P$ ($k \in \mathbb{N}$, $P \in X$). Induction

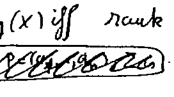
Step 3 If $D \in Div(X)$, $\deg D = 0 \Rightarrow D = D_+ - D_-$. Apply ↑ to D_+, D_- ...

(Def) A var X called alg gr if we are given (mor) $\mu : X \times X \to X$, $i : X \to X$ satisfying axioms...

(Ex) $\mathbb{A}^1$ & $\mathbb{A}^1 \setminus \{0\}$ alg gr with $+$, $\cdot$ resp. Call them $\mathbb{G}_a, \mathbb{G}_m$. (Ex) $GL(n)$, $SL(n)$.

(Prop) Let X be the cubic above (1), $P_0 \in X$, then the group law transported from $Cl^0(X)$ to X gives X a str. of alg gr with 0-elt P_0.

Pf. Step 1 We describe the gr law $(P, Q) \mapsto P \oplus Q$ on X geometrically. We have $P \oplus Q = S$ iff $P - P_0 + Q - Q_0 \sim S - Q_0$ i.e iff $P + Q \sim S + Q_0$ so S is given by fig. (*). Next $P' = \ominus P$ is given by $P - P_0 + P' - P_0 \sim 0$ or by $P + P' \sim 2P_0$ hence by



Step 2. Derive formulae for $\oplus, \ominus$ (rational exprs of coord). (2)

E
NP (Th) The only ~~[illegible]~~ alg gr of dim 1 are $\mathbb{G}_a, \mathbb{G}_m$ and $Z(x_2^2 x_0 - \Phi)$ [all comm !]

One can gener. facts on $T_P X$ to space curves:

(Def) X ~~[illegible]~~ any variety, $P \in X$ (3). Define $T_P X = \mathrm{Hom}(m/m^2, k)$ (call it tang. sp (4)). So $P \notin Sing(X)$ iff $\dim T_P X = 1$. Define $T_P^* X = m/m^2$ (cotg sp)

(Prop) $X \subset \mathbb{A}^n$ any var, $I(X) = (f_1, \dots, f_m) \subset k[y_1, \dots, y_n]$, $P \in X$, $P = (a_1, \dots, a_n)$ $\Rightarrow \frac{m}{m^2} = (k\bar{y}_1 + \dots + k\bar{y}_n)/\Sigma$ where Σ space gen by $\sum_j \frac{\partial f_1}{\partial y_j}(P)(y_j - a_j), \sum_j \frac{\partial f_2}{\partial y_j}(P)(y_j - a_j), \dots, \sum_j \frac{\partial f_m}{\partial y_j}(P)(y_j - a_j)$ [So if X curve, $P \notin Sing(X)$ iff rank $\left(\frac{\partial f_i}{\partial y_j}(P)\right) = 1$]

Pf. Similar to that in Lect 3 for case $n = 2$, $m = 1$. ~~[illegible]~~

(Rem) $T_P \mathbb{A}^n = (ky_1 + \dots + ky_n)^* \hookleftarrow T_P X$.

(Def) $X \subset \mathbb{P}^n$ non-sing curve ($X = Z(F_1, \dots, F_m)$), $P \in X$. Def the proj tg line $\overline{T_P X} \subset \mathbb{P}^n$ as $\overline{T_P X} = Z\left(\sum_j \frac{\partial F_1}{\partial x_j}(P) x_j, \dots, \sum_j \frac{\partial F_m}{\partial x_j}(P) x_j\right)$

(Prop) $X \subset \mathbb{P}^n$ non-sing curve, $Q \in \mathbb{P}^n \setminus X$, $P \in X$, $\pi : X \hookrightarrow \mathbb{P}^n \setminus Q \to \mathbb{P}^{n-1}$ the proj. Then $T_P \pi : T_P X \to T_{\pi(P)} \mathbb{P}^{n-1}$ is inj iff $Q \notin \overline{T_P X}$ [Pf. May ass $P = (1:0:\dots:0)$, $Q = (0:1:\dots:0) \Rightarrow$ proj π is given by $(y_1, \dots, y_n) \mapsto (y_2, \dots, y_n)$ & compute]

(Prop) Let $f : X \to Y$ finite injective & $T_P f$ inj $\forall P$. $\Rightarrow f$ embedding (:= isom of X with $f(X)$).

Pf. May ass Y aff $\mathcal{O}(Y) = A$, $\mathcal{O}(X) = B$. Suff $A \to B$ surj (Lect 6, (α)). Suff $A_m \to B_m$ surj $\forall$ max m (Lect 1). Note B_m local (f inj !) and finite $/A_m$ so we are reduced to ~~[illegible]~~ apply to $A_m \to B_m$:

(L) $f : A \to B$ morph of loc noeth rings $m_B \cap A = m_A$, $\frac{A}{m_A} \cong \frac{B}{m_B}$, $\frac{m_A}{m_A^2} \twoheadrightarrow \frac{m_B}{m_B^2}$, B is finite $/A$. Then f is surj

Pf. We have $m_A B \subset m_B$. Nak' for the B-module m_B: $m_A B$, gen m_B/m_B^2 ~~[illegible]~~ $\Rightarrow m_A B = m_B$. Now Nak' for the A-module B: 1 gen $B/m_A B$ ($= B/m_B = A/m_A$ as A-mod) $\Rightarrow$ 1 gen B as A-module $\Rightarrow$ qed//.

(C) Any non-sing curve $X \subset \mathbb{P}^n$ can be embedded (via proj) in $\mathbb{P}^3$ [use ↑ + Lect 7].

(4) If $f : X \to Y$ morph of var, $P \in X$ then we have an induced lin map $T_P f : T_P X \to T_{f(P)} Y$.

(3) m = max of $\mathcal{O}_{X,P}$.

(2) Take $y = mx + n \ni P = (a_1, b_1)$, $Q = (a_2, b_2)$; substitute in $y^2 = \varphi(x) \Rightarrow$ equs of 3rd degree with 2 known roots a_1 & a_2. The 3rd root is (coeff. $- a_1 - a_2$) o.s.o.

(1) everything holds for $x^3 - x$ replaced by any $\varphi \in k[x]$, $\deg \varphi = 3$, φ with distinct roots

(so $X = Z(x_2^2 x_0 - \Phi)$, where $\Phi(x_0, x_1, x_2) = x_0^3 \varphi\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$ with same proofs

Lecture 11. Differentials.

Their set $Der_A(B,M)$

A ring, B an A-alg, M a B-mod.

(Def) An A-der $d: B \to M$ means d additive & $d(bb') = b\,db' + b'\,db$ & $da = 0$ $\forall a \in A$.

(Def) A module of rel diff for B/A means a pair $(\Omega_{B/A}, d)$, $d: B \to \Omega_{B/A}$ A-deriv s.t. $\forall d': B \to M$ another A-der $\exists!$ B-mod map $\Omega_{B/A} \xrightarrow{B\text{-mod}} M$ making ... [clearly $(\Omega_{B/A}, d)$ unique up to iso]

(Prop) $(\Omega_{B/A}, d)$ always exists [Pf: F = free B-mod gen by $\{db \mid b \in B\}$, put $\Omega_{B/A} = F/F_1$, F_1 = B-mod gen by $d(b+b') - db - db'$, $d(bb') - \dots$, da.]. So $\Omega_{B/A}$ is gen by $\{db \mid b \in B\}$.

(β) (Prop) $\Omega_{S^{-1}B/A} \simeq S^{-1}\Omega_{B/A}$ ($\forall S \subset B$ mult) [Pf. univ]

(Prop) $\Omega_{A[x_1,\dots,x_n]/A} = \bigoplus_{i=1}^{n} A\,dx_i$ [with univ!]; (Cor) If B = loc of a f.g. A-alg $\Rightarrow$ $\Omega_{B/A}$ f.g. B-mod.

(α) (Prop) $I \subset B$ ideal (B polyn), $C = B/I$ $\Rightarrow$ $I/I^2 \xrightarrow{d} \Omega_{B/A}/I\Omega_{B/A} \to \Omega_{C/A} \to 0$ exact [Pf. Check $\frac{\Omega_{B/A}}{I\Omega_{B/A} + \langle dI \rangle}$ satisfies the univ prop of $\Omega_{C/A}$].

(δ) (Prop) K/k field ext f.g., k alg cl $\Rightarrow$ $\exists x_1, \dots, x_n$ transc basis [explain!] s.t. $\Omega_{K/k} = \bigoplus K\,dx_i$ [universality; use $K = k(x_1,\dots,x_n,\xi)$, $f(\xi) = 0$, $\frac{\partial f}{\partial x_{n+1}}(\xi) \neq 0$ & put $dy = -\left(\frac{\partial f}{\partial x_{n+1}}(\xi)\right)^{-1} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\xi)\,dx_i$].

(γ) (Prop) X aff var, $B = \mathcal{O}_{X,P}$, m = max of $\mathcal{O}_{X,P}$ $\Rightarrow$ $\frac{m}{m^2} \xrightarrow{d} \Omega_{B/k}/m\Omega_{B/k}$ iso.

Pf: By (α) surjectivity. Now we want $Hom_k(\Omega_{B/k}/m\Omega_{B/k}, k) \to Hom_k(\frac{m}{m^2}, k)$ surj. But $Hom_k(\Omega_{B/k}/m\Omega_{B/k}, k) \overset{easy}{=} Hom_B(\Omega_{B/k}, k) = Der_k(B,k)$. Let $\varphi: \frac{m}{m^2} \to k$ and define for $f \in B$, $\tilde{\varphi}(b) = \varphi(b - b(P)) \in k$ where $b(P)$ = im of b in B/m. Check $\tilde{\varphi} \in Der_k(B,k)$!

(ε) (Prop) A noeth local integral dom, m = max ideal, $k = \frac{A}{m}$, $K = Q(A)$ ($S^{-1}A$ ($S = A \setminus \{0\}$)). Let M a f.g. A-mod. If $\dim_k \frac{M}{mM} = \dim_{Q(A)} S^{-1}M = r$ $\Rightarrow$ M free of rank r.

Pf. Nak $\Rightarrow$ M gen by r elts $\Rightarrow$ $\exists\ \varphi: A^r \to M$ surj so $\exists$ $0 \to R \to A^r \to M \to 0$ $\overset{Lect\ 1}{\Rightarrow}$ $0 \to S^{-1}R \to Q(A)^r \to S^{-1}M \to 0$ $\Rightarrow$ $S^{-1}R = 0$. Since R is torsion free $\Rightarrow$ $R = 0$.

(Def) Let X be an affine variety. Define $\Omega(X) := \Omega_{\mathcal{O}(X)/k}$ and $\Omega_{X,P} := \Omega_{\mathcal{O}_{X,P}/k}$ ($\overset{β}{=} \Omega(X)_{m_P}$). Call them: $\Omega(X)$ = reg diff on X, $\Omega_{X,P}$ = diff on X reg at P.

(Prop) Let X be a non-sing affine curve. Then $\forall P \in X$, $\Omega_{X,P}$ is a free $\mathcal{O}_{X,P}$-mod of rank 1 gener. by dt_P [(t_P) = max of $\mathcal{O}_{X,P}$] so $\Omega_{X,P} = \mathcal{O}_{X,P}\,dt_P$.

Pf. Let $B = \mathcal{O}_{X,P}$. By (γ) $\dim_k \Omega_{B/k}/m\Omega_{B/k} = 1$. By (δ) $\dim_{Q(B)} S^{-1}\Omega_{B/k} = 1$. Conclude by (ε) (Last ass from Nak & (α)!)

(Cor) Let X be non-sing affine curve. Then $\Omega(X)$ is torsion free (hence $\subset \Omega_{K(X)/k}$) and $\Omega(X) = \bigcap_{P \in X} \Omega_{X,P}$ ($\cap$ in $\Omega_{K(X)/k}$!) [Pf. Trivial verif, see Lect 4 & Lect 6].

② explain |

① = a substitute for integrals $\int g(x, u(x))\,dx$, u alg fun! Intuitively we study abstract quantities $g(x,u)\,dx$.

(Def) X a non-sing curve. Define $\Omega(X) = \{\omega \in \Omega_{K(X)/k} \mid \omega \in \Omega_{X,P}, \forall P \in X\}$. By above prop this dfn agrees with the previous for affine X. Define genus $g(X) = g = \dim_k \Omega(X)$ (*).

(Prop) Let $X = Z(f) \subset \mathbb{A}^2$ be non-sing curve, $A = k[x,y]/(f) = k[\bar{x},\bar{y}]$. Then $\dfrac{d\bar{x}}{\partial f/\partial y} = -\dfrac{d\bar{y}}{\partial f/\partial x} \in \Omega_{K(X)/k}$ defines a differential $\omega_f \in \Omega(X)$.

$\underset{\Omega(X_{\partial f/\partial y})}{\cap}$ $\quad \underset{\Omega(X_{\partial f/\partial x})}{\cap}$

(because recall for $(0,0)$: $m/m^2 = \dfrac{kx+ky}{k(ax+by)}$, $a = \dfrac{\partial f}{\partial x}(0)$, $b = \dfrac{\partial f}{\partial y}(0)$)

Since by Lect 4 $\bar{x} - \bar{x}(P)$ is a par at (any) $P \in X_{\partial f/\partial y}$ (and analogously for $\bar{y}$) $\Rightarrow \mathcal{O}_{X,P}\, \omega_f = \Omega_{X,P}$ for all P $\Rightarrow \Omega(X) = \mathcal{O}(X)\omega_f$ (i.e. $\Omega(X)$ free of rk 1 gen by ω_f!); call ω_f the nat diff.

(Example) Let $X = Z(y^2 - \varphi(x)) \subset \mathbb{A}^2$, $\varphi \in k[x]$, $\deg \varphi = 3$, φ with dist roots. Then $\Omega(X)$ is gen by $\omega = \dfrac{d\bar{x}}{\bar{y}} = -\dfrac{2d\bar{y}}{\varphi'(\bar{x})}$ (this has to do with the "elliptic integral" $\int \dfrac{dx}{\sqrt{\varphi(x)}}$)

(Def) Let X be a non-sing (proj) curve & $\underset{0}{\overset{+}{\omega}} \in \Omega_{K(X)/k}$. Define $(\omega) \in \mathrm{Div}(X)$ by $(\omega) = \sum v_P(f_P)P$ where $\omega = f_P\, dt_P$ [t_P par at P; it is well def bec if $t = ut'$, $u \in \mathcal{O}^*_{X,P}$ and we write $du = h\,dt'$, $h \in \mathcal{O}_{X,P}$ $\Rightarrow \omega = f\,dt = f(u + t'h)dt'$ & $u + t'h \in \mathcal{O}^*_{X,P}$]. We have clearly 1) $(f\omega) = (f) + (\omega)$ $\forall f \in K(X)^*$, $\forall \underset{0}{\overset{+}{\omega}} \in \Omega_{K(X)/k}$ and 2) $(\omega) \geq 0$ iff $\omega \in \Omega(X)$.

Since $\dim_{K(X)} \Omega_{K(X)/k} = 1$, $\forall \underset{0}{\overset{+}{\omega}}, \underset{0}{\overset{+}{\omega'}} \in \Omega_{K(X)/k}$, $\exists f \in K(X)^*$ s.t. $\omega' = f\omega \Rightarrow (\omega) \sim (\omega')$!

Call each $(\omega) \in \mathrm{Div}(X)$ a can. divisor & their class in $Cl(X)$ the can. class K_X.
We denote by K_X any can. divisor (ω).

(Ex) If $X = \mathbb{P}^1$; coord x_0, x_1, $y = \dfrac{x_1}{x_0}$ (coord on $\mathbb{P}^1 \setminus Z(x_0)$). Take $\omega = dy$. Let $z = \dfrac{x_0}{x_1}$ coord on $\mathbb{P}^1 \setminus \underset{P_\infty}{Z(x_1)}$ $\Rightarrow z = \dfrac{1}{y} \Rightarrow dz = -\dfrac{dy}{y^2}$ so $(\omega) = -2 \cdot P_\infty$ so $\deg(\omega) = -2$ so $g(X) = 0$ and $K_X \sim -2P$ (P any point!).

(Prop) Let $X = Z(F) \subset \mathbb{P}^2$ be a non-sing curve of degree d and let $H = X \cdot L \in \mathrm{Div}(X)$ where L is a line $\subset \mathbb{P}^2$. Then $K_X \sim (d-3)H$ ~~$g(X) = \frac{(d-1)(d-2)}{2}$~~ (in partic for different degrees $\overset{2)}{\Rightarrow}$ non-isom curves, bec. $\deg K_X$ is an invariant!)

Pf. Put $U_i = \mathbb{P}^2 \setminus Z(x_i)$, $y_1 = \dfrac{x_1}{x_0}$, $y_2 = \dfrac{x_2}{x_0}$, $f(y_1,y_2) = F(1,y_1,y_2)$. Let ω_f be the nat diff ass to f: $\Omega(X \cap U_0) \subset \Omega_{K(X)/k}$

$\omega_f = \dfrac{dy_1}{\partial f/\partial y_2} = -\dfrac{dy_2}{\partial f/\partial y_1} \Rightarrow (\omega_f) \cap U_0 = 0$ (no "poles" or "zer" on U_0). Put $z_1 = \dfrac{x_0}{x_1}$, $z_2 = \dfrac{x_2}{x_1}$ coord on U_1 $\Rightarrow z_1 = \dfrac{1}{y_1}$, $z_2 = \dfrac{y_2}{y_1}$ $\Rightarrow$ By Lect 10, $\partial f/\partial y_2 = \dfrac{\partial F}{\partial x_2}(1,y_1,y_2)$ and $\partial g/\partial z_2 = \dfrac{\partial F}{\partial x_2}(z_1,1,z_2) = \dfrac{\partial F}{\partial x_2}\left(\dfrac{1}{y_1},1,\dfrac{y_2}{y_1}\right) = \dfrac{\partial F}{\partial x_2}\left(\dfrac{1}{y_1},\dfrac{y_1}{y_1},\dfrac{y_2}{y_1}\right) \underset{\text{of deg } d-1}{\overset{\text{hom}}{=}}$

$= \dfrac{1}{y_1^{d-1}} \dfrac{\partial F}{\partial x_2}(1,y_1,y_2) = z_1^{d-1}(\partial f/\partial y_2)$. Since $dy_1 = -\dfrac{dz_1}{z_1^2} \Rightarrow \omega_f = -z_1^{d-3}\dfrac{dz_1}{\partial g/\partial z_2}$

$\Rightarrow \omega_f = -z_1^{d-3}\omega_g \Rightarrow (\omega_f) \sim (d-3)H$ (3). Now we compute $\Omega(X) = \Omega(X \cap U_0) \cap \Omega(X \cap U_1)$.

~~So we look for those $P \in k[y_1,y_2]$ s.t. $\exists Q \in k[z_1,z_2]$ s.t. $P(y_1,y_2)\omega_f = Q(z_1,z_2)\omega_g$ ($= Q(z_1,z_2)z_1^{3-d}\omega_f$) $\Leftrightarrow P(y_1,y_2) = -z_1^{3-d}Q(z_1,z_2) \Leftrightarrow P(y_1,y_2)/y_1^{d-3} = -Q(\frac{1}{y_1},\frac{y_2}{y_1}) \Leftrightarrow \deg P \leq d-3$. Two different P's yield diff elts in $\mathcal{O}(X \cap U_0) = k[y_1,y_2]/(f)$ because if P_1 & P_2 are 2 pol of deg $\leq d-3$ s.t. $P_1 - P_2 \in (f) \Rightarrow P_1 = P_2$. The space of P's as above has dim $(d-1)(d-2)/2$. QED.~~

(*) apriori it may be ∞; we shall see $g < \infty$ if X proj!

Put $g(z_1,z_2) = F(z_1,1,z_2)$ so $Z(g) = X \cap U_1$

3) if we assume, as we may that $X \subset U_0 \cup U_1$ ($\Leftrightarrow (0:0:1) \notin X$)

Lecture 12 Linear systems and Riemann-Roch theorem

(Def) Let X be a non-sing proj curve & $D \in \mathrm{Div}(X)$. Define $L(D) = \{f \in K(X)^* \mid (f) + D \geq 0\} \cup \{0\}$ (lin sp. ass to D)
(If $D = \sum n_P P$, $L(D) = \{f \in K(X)^* \mid v_P(f) \geq -n_P, \forall P \in X\}$). It is a k-lin space.
Define $\ell(D) = \dim_k L(D)$ (dim of D) (a priori ∞). Define $|D| = \{E \in \mathrm{Div}(X) \mid E \sim D, E \geq 0\}$ (= complete lin sys ass to D)
Note $\ell(0) = 1$.

(Prop) $D_1 \sim D_2 \Rightarrow L(D_1) \simeq L(D_2)$ [Pf if $D_1 - D_2 = (g)$ then $L(D_1) \xrightarrow{\sim} L(D_2)$, $f \mapsto f \cdot g$]
so $\ell(D_1) = \ell(D_2)$.

(Prop) $|D| = \mathbb{P}(L(D)) := (L(D) \setminus \{0\})/\sim$ where $\sim$ (so $|D| \neq \phi$ iff $\ell(D) \geq 1$) [Consider map $L(D) \setminus \{0\} \to |D|$,
$f \mapsto D + (f)$; surj clear. To compute fibres ass $D + (f) = D + (g) \Rightarrow (fg^{-1}) = 0 \Rightarrow fg^{-1} \in \mathcal{O}(X)^* = k^*$ Lect 6.]

(Prop) $\ell(D) < \infty$ [so $|D| \simeq \mathbb{P}^{r-1}$ where $r = \ell(D)$] more precisely $\ell(D) \leq \deg D + 1$
Pf (we may ass $D \geq 0$) Let $D = \sum n_P P$ & consider the k-lin map $\varphi : L(D) \to \bigoplus \dfrac{t_P^{-n_P} \mathcal{O}_P}{\mathcal{O}_P} \simeq \bigoplus \underbrace{\dfrac{\mathcal{O}_P}{t_P^{n_P} \mathcal{O}_P}}_{\text{fin dim}}$
($\varphi = \oplus \varphi_P$, $\varphi_P(f) = f$); $\ker \varphi \subset \mathcal{O}(X) = k$; $\Rightarrow \dim L(D) \leq \deg D + 1$.

(Cor) The genus $g(X) < \infty$ [Pf: Ass $K_X = (\omega)$ for $\omega \in \Omega_{K(X)/k}$. Then we have (Lect 3) an
iso $\Omega(X) \xleftarrow{\sim} L(K_X)$, $f\omega \leftarrow f$]. Moreover $g(X) \leq \deg K_X + 1$ if $g \neq 0$. (*) Note $g(X) = \ell(K_X)$.

(α) (Rem) $\deg D < 0 \Rightarrow |D| = \phi \Rightarrow \ell(D) = 0$. Moreover if $\deg D = 0$ and $\ell(D) \geq 1 \Rightarrow D \sim 0$.

BASIC (Th) (Riem-Roch) $\ell(D) - \ell(K - D) = \deg D + 1 - g$ [To be proved later. First - applications]

(Cor) $\deg K = 2g - 2$ [Pf $D = K$; $\ell(0) = 1$] (Cor) $\deg D \geq g \Rightarrow |D| \neq \phi$. [obv] (**)

(Cor) genus of plane curve $= (d-1)(d-2)/2$.

(β) (Cor) $\deg D \geq 2g - 1 \underset{(\alpha)}{\Rightarrow} \ell(K - D) = 0 \Rightarrow \ell(D) = \deg D + 1 - g$ [Pf $E = K - D \Rightarrow \deg E < 0 \overset{?}{\Rightarrow} \ell(E) = 0$]

(Cor) $g(X) = 0 \Leftrightarrow X \simeq \mathbb{P}^1$ [„$\Leftarrow$" cf Lect 11; „$\Rightarrow$" take $P, Q \in X, P \neq Q$. RR for
$D = P - Q \Rightarrow \ell(P - Q) = 1 \underset{(\alpha)}{\Rightarrow} P - Q \sim 0 \underset{\text{Lect 10}}{\Rightarrow} X \simeq \mathbb{P}^1$].

(Cor) $g(X) = 1 \Leftrightarrow K_X \sim 0$ [„$\Leftarrow$" by $g = \ell(K)$; „$\Rightarrow$" $\ell(K) = 1$ &
$\deg K_X = 2g - 2 = 0 \underset{(\alpha)}{\Rightarrow} K_X \sim 0$]

(Def) By a lin sys we understand a lin. subspace Λ of some $|D|$ ($\simeq \mathbb{P}^r$); Λ is
called complete if $\Lambda = |D|$ (in any case $\Lambda \simeq \mathbb{P}^s$ for some s).
$P \in X$ is called a base point for Λ if $P \in \mathrm{Supp}\, D$, $\forall D \in \Lambda$. Call $Bs(\Lambda) = \ldots$
Λ called (base pt) free if $Bs(\Lambda) = \phi$. (Eg. on $\mathbb{P}^1$, $P \in \mathbb{P}^1 \Rightarrow |P|$ is free
while on a nonsing cubic X in $\mathbb{P}^2$, $P \in X \Rightarrow |P| = \{P\}$ is not free for $\{P\} = Bs|P|$;
If $X \subset \mathbb{P}^n$ & $H \subset \mathbb{P}^n$ hyperplane $\Rightarrow |X \cdot H|$ is free).

(γ) (Prop) $\forall P \in X$, $\ell(D) \geq \ell(D - P) \geq \ell(D) - 1$. $P \in Bs|D|$ iff $\ell(D - P) = \ell(D)$.
So by (β) if $\deg D \geq 2g \Rightarrow |D|$ free
Pf. Let $D = \sum n_P P$. $\exists$ lin map $L(D) \to \dfrac{t_P^{-n_P} \mathcal{O}_P}{t_P^{-n_P+1} \mathcal{O}_P} \simeq \dfrac{\mathcal{O}_P}{t_P \mathcal{O}_P}$ [$f \mapsto f$ mod ... $\xrightarrow{\sim}$ $t^{n_P} f$ mod ...]
whose ker is $L(D - P)$. 2nd stat clear.

(**) (Cor) (generalizing the sit from cubic) Let $P_0 \in X$. Then for any $D \in \mathrm{Div}(X)$ ($\deg D = 0$), $D \sim P_1 + \cdots + P_g - g P_0$ for some $P_1, \ldots, P_g$
Indeed $\deg(D + g P_0) = g \Rightarrow |D + g P_0| \neq \phi$. Take $P_1 + \cdots + P_g \in |D + g P_0|$.

(*) In fact $g(X)$ & $\deg K_X$ are rel. in a more precise way (see later) $\deg K_X = 2g(X) - 2$.

(Def) X non-sing proj curve, $f: X \to \mathbb{P}^n$ morph. called non-deg if $f(X) \not\subset$ hyperplane.
Let $H \subset \mathbb{P}^n$ hyperplane. Def $|H|$ = set of hyperplanes in $\mathbb{P}^n$ ($|H| = \check{\mathbb{P}}^n \cong \mathbb{P}^n$ Lect. 5)
Define $f^*H \in \mathrm{Div}(X)$ exactly as $X \cdot H$ in case f emb. (Lect 9). Exactly as in Lect 9 $\Rightarrow f^*H_1 \sim f^*H_2$ [their diff $= (\frac{H_2}{H_1})$] so $\Rightarrow$ map $f^*: |H| \to |f^*H|$; its image $\Lambda = f^*|H|$ is a (free) lin sys (not nec. complete). If it is complete, f called lin normal (or X).

(8) (Prop) $\exists$ bij bet $\{$nondegenerate morph $X \to \mathbb{P}^n\}/\sim^{(*)}$ and $\{$free lin. systems on $X\}$. Two morph $f_1: X \to \mathbb{P}^{n_1}$, $f_2: X \to \mathbb{P}^{n_2}$ are obt. from each other by proj $\mathbb{P}^{n_2} \dashrightarrow \mathbb{P}^{n_1}$ (3) iff the lin sys $\Lambda_2 \supset \Lambda_1$.
In partic $f: X \to \mathbb{P}^n$ lin normal iff $\nexists\ \tilde{f}: X \to \mathbb{P}^N$ & a proj $\pi: \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ s.t. $\pi\tilde{f} = f$ (2)

Pf. Let $f: X \to \mathbb{P}^n$ be nondeg; associate $\Lambda := f^*|H|$ as above. Note $\dim \Lambda = n$ [for if $\underbrace{D = f^*H}_{(H = Z(L))}$ & $L(H) = \{f \in K(\mathbb{P}^n) \mid f = \frac{L_1}{L},\ L_1 \in S_1 = \sum k x_i\}$. then we have a lin map $\mathrm{res}: L(H) \to L(D)$; it is inj by non-deg $\Rightarrow$ induces $\mathbb{P}(L(H)) = |H| \to \mathbb{P}(L(D)) = |D|$ inj]
Conversely given Λ free, $\Lambda \subset |D| \overset{D \geq 0}{\Rightarrow} \Lambda = \mathbb{P}(V)$ with $V \subset L(D)$ pick $f_0, \dots, f_n$ a k-basis of V & define $f = f_\Lambda: X \setminus D \to \mathbb{P}^n$, $f(P) = (f_0(P) : \dots : f_n(P)) \Rightarrow f$ everywhere def by Lect 9
E (for $Q \in \mathrm{Supp}\, D$, if $D = \sum n_Q Q$, $f(Q) = ((t_Q^{n_Q} f_0)(Q), \dots, (t_Q^{n_Q} f_n)(Q))$!). One checks the two assoc. give bij (4) (Note) $|D| = \mathbb{P}(L(D))$ identifies with $\mathbb{P}(L(H)) = \check{\mathbb{P}}^n$ so $\mathbb{P}^n$ identif with $|D|^\vee$.

(Def) A (free) lin sys Λ separates pts if $\forall P, Q \in X$, $P \neq Q$, $\exists D \in \Lambda$ s.t. $P \in \mathrm{Supp}\, D$, $Q \notin \mathrm{Supp}\, D$.
It separates tangent vectors if $\forall P \in X$ $\exists D \in \Lambda$ s.t. P appears with $n_P = 1$ in D (5)

(Prop) $|D|$ (is free) separates pts & Tg vectors provided $\forall P, Q \in X$, $\ell(D-P-Q) = \ell(D) - 2$. Moreover in this case $f_{|D|}: X \to \mathbb{P}^n$ is an embedding (6)
Pf. $\ell(D-P-Q) = \ell(D)-2\ \forall P,Q \Rightarrow \ell(D-P) = \ell(D)-1\ \forall P \Rightarrow |D|$ free. Hyp $\Rightarrow D-P$ is free $\forall P$. $\Rightarrow$
$\Rightarrow |D|$ separates pts & Tg vectors. Now "$|D|$ separates pts" $\Rightarrow$ "$f_{|D|}$ injective" (for if $P \neq Q$,
$R = f_{|D|}(P) = f_{|D|}(Q) \Rightarrow \forall$ hyperplane $H \subset \mathbb{P}^n$ either passes through R (in this case $P, Q \in \mathrm{Supp}\, f^*H$) or $P \notin H$ (in this case $P, Q \notin \mathrm{Supp}\, f^*H$; since $f^*|H| = |D|$ ↯)
Claim: "$|D|$ sep tg vect" $\Rightarrow$ "$T_P f_{|D|}$ inj $\forall P \in X$" (and we are done by Lect 10).
; $T_P f_{|D|}$ inj $\Leftrightarrow \frac{m_{f(P)}}{m^2_{f(P)}} \to \frac{m_P}{m^2_P}$ surj. Assume the latter is not surj $\Rightarrow$ the latter is the zero map $\Rightarrow \forall$ hyperplane H passing through $f(P)$ we have $v_P(f^*H) \geq 2$ ↯.

(Cor) $\deg D \geq 2g+1 \Rightarrow f_{|D|}$ embedding (i.e. $|D|$ very ample (6))
(Cor) $\forall P \in X \Rightarrow nP$ is very ample for $n \geq 2g+1$
(Cor) $\forall P \in X \Rightarrow X \setminus P$ affine [$\forall D$ very ample $X \setminus \mathrm{Supp}\, D$ affine !].

(4) If $H = Z(\sum_{i=0}^n \alpha_i x_i) \Rightarrow f^*H = (\sum_{i=0}^n \alpha_i f_i) + D$ | (5) i.e. $v_P(D) = 1$ | (6) If $f_{|D|}$ emb we say D is very ample.
(3) proj from a lin space $L \subset \mathbb{P}^{n_2}$ s.t. $L \cap f_2(X) = \emptyset$.
(2) Moreover under this corresp, if $f: X \to \mathbb{P}^n$ corr to Λ then $\dim \Lambda = n$ (as a proj space)
(*) $f: X \to \mathbb{P}^n$, $g: X \to \mathbb{P}^m$ called equiv if $n = m$ & $\exists$ proj transf $\sigma: \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ $\sigma g = f$

Lecture 13. Hurwitz theorem. Elliptic curves ~~Hyperell. & [illegible]~~

(Def) Let $f: X \to Y$ be a non-constant morph of non-sing proj curves & $P \in X$. Define $e_P = v_P(t_Q)$ where $Q = f(P)$, t_Q par at $\mathcal{O}_Q$ (so $f^*Q = \sum_{f(P)=Q} e_P \cdot P$). P called ramif pt if $e_P > 1$.

(Prop) f as ↑. Then $K_X \sim f^*K_Y + \sum_{P \in X} (e_P - 1)P$. In partic $\exists$ only fin many ramif (pts).

Pf. $\exists$ inj map $f^*: \Omega_{K(Y)/k} \to \Omega_{K(X)/k}$ (Lect 11). Let $\omega \in \Omega_{k(Y)/k}$, $\omega = u_Q t^{n_Q} dt_Q$ ($Q \in Y$, $u \in \mathcal{O}_Q^*$) so $K_Y = (\omega) = \sum_Q n_Q Q$ so $f^*K_Y = \sum_{\substack{f(P)=Q \\ Q \in Y}} n_Q e_P \cdot P$. Now if $t_Q = \tilde{u}_P t_P^{e_P}$, $\tilde{u}_P \in \mathcal{O}_P^*$ $\Rightarrow$ $f^*\omega = u_Q \tilde{u}_P t_P^{e_P n_Q} d(\tilde{u}_P t_P^{e_P}) = u_Q \tilde{u}_P t_P^{e_P n_Q} [t_P^{e_P} d\tilde{u}_P + \tilde{u}_P e_P t_P^{e_P - 1} dt_P]$
$= \underbrace{w_P}_{\in \mathcal{O}_P^*} t_P^{e_P n_Q + e_P - 1} dt_P \Rightarrow$ qed//

(Cor) f as ↑ $\Rightarrow$ $2g(X) - 2 = (\deg f)(2g(Y) - 2) + \sum (e_P - 1)$ [Pf: take degrees]. In partic $g(X) \geq g(Y)$. In partic (Lüroth) ($X = \mathbb{P}^1 \Rightarrow Y \simeq \mathbb{P}^1$) i.e. unirat $\Leftrightarrow$ rat; comments about higher dim.

(Cor) $\mathbb{P}^1$ is simply conn [$\overset{\text{def}}{=}$ any étale cov is trivial; here ét cov means "no ram. pts"].

(L1) X elliptic, $P, Q \in X$. $\Rightarrow |P + Q|$ is free of dim 1 (so defines $f: X \xrightarrow{2} \mathbb{P}^1$ by Lect 12)

Pf. RR + ① in Lect 12.

(L2) Let $f_1: X \xrightarrow{2} \mathbb{P}^1$, $f_2: X \xrightarrow{2} \mathbb{P}^1$ of deg 2 ($f_i = f_{|2P_i|} \Rightarrow$) $\exists$ aut σ, τ s.t.
$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ f_1 \downarrow & & \downarrow f_2 \\ \mathbb{P}^1 & \xrightarrow{\tau} & \mathbb{P}^1 \end{array}$$

Pf. Let $f_3 = f_{|P_1 + P_2|}: X \xrightarrow{2} \mathbb{P}^1$ and $\sigma: X \to X$ interch. fibres (*) so $\sigma(P_1) = P_2$ so $\sigma^*|2P_2| = |2P_1| \Rightarrow \exists \tau$ (Lect 12, [illegible])

(L3) Let $f: X \xrightarrow{2} \mathbb{P}^1$ ($f = |2P_0|$) $\Rightarrow$ $\exists$ exactly 4 ram pts with dist images $Q_1, Q_2, Q_3, Q_4 \in \mathbb{P}^1$. [$\Leftarrow$ Hur]. Take any $\sigma \in \mathbb{P}^1$ s.t. $\sigma(\{Q_1, Q_2, Q_3, Q_4\}) = \{0, 1, \lambda, \infty\}$; $\lambda \in k = \mathbb{A}^1$ and put $j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$. Then $j(\lambda)$ depends only on X (not on f or σ!). Call it $j(X)$ (the j-inv of X!)

Pf. Let λ' corr to another choice (f', σ'). By ↑ $\exists$ $T \in PGL(2)$ s.t. $T(\{0, 1, \lambda, \infty\}) = \{0, 1, \lambda', \infty\}$.

Claim (This happens iff) λ and λ' and conj under the action on $k \setminus \{0, 1\}$ of the group $\Sigma \subset \mathrm{Aut}(k \setminus \{0,1\})$ generated by σ_1, σ_2 where $\sigma_1(t) = \frac{1}{t}$, $\sigma_2(t) = 1 - t$. Now check (trivial) that $j(\frac{1}{t}) = j(t) = j(1 - t)$ $\forall t$.

(L3) X ell, $P_0 \in X$. Then $|3P_0|$ gives an emb $u: X \to \mathbb{P}^2$ [$\ell(3P_0) = 3$ by RR & emb $\Leftarrow$ $\deg 3P_0 \geq 2g + 1$]. Moreover $\exists$ coordinates x_0, x_1, x_2 on $\mathbb{P}^2$ s.t. if $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$ then $u(X) \cap [\mathbb{P}^2 \setminus Z(x_0)] = Z(y^2 - x(x-1)(x-\lambda))$ where $\lambda \in k \setminus \{0,1\}$ is the same with the one in prec. lemma (mod action of Σ) and $u(P_0) = (0:1:0)$ is an inflection point of $u(X)$ [in which $Z(x_0)$ is tangent to $u(X)$]

Pf. RR $\Rightarrow$ $\ell(nP_0) = n$. Consider $k = L(P_0) \subset L(2P_0) \subset L(3P_0) \subset \cdots$. Take $x \in L(2P_0) \setminus L(P_0)$, $y \in L(3P_0) \setminus L(2P_0)$ (**). $\forall$ polyn $F \in k[t_1, t_2]$ s.t. $F(x, y) = 0 \Rightarrow$ $\mathrm{Im}\, u \subset Z(F)$ (**). Now $\mathrm{Im}\, u \not\subset$ line or conic since they are rat. Now $1, x, y, x^2, xy, x^3, y^2 \in L(6P_0) \Rightarrow$ they are lin dep. The 1st 5 are lin ind ($\mathrm{Im}\, u \not\subset$ conic) $\Rightarrow$ $\exists$ lin comb of ↑ in which either x^3 or y^2 occur. They both occur since $v_{P_0}(x^3) = v_{P_0}(y^2) = -6$ & $v_{P_0}(\text{others}) \geq -5$

may ass $\exists$ rel $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$; $y' = y + \frac{1}{2}(a_1 x + a_3)$ $\Rightarrow$ may ass $y^2 = \varphi(x)$, φ cubic. May ass $\varphi = x(x-1)(x-\lambda)$. λ is the one above [consider proj from $(0:1:0)$ onto x-axis]

(**) so $1, x, y$ is a basis of $L(3P_0)$ so $u_{|3P_0|}: X \to \mathbb{P}^2$ is given by $u(P) = (1 : x(P) : y(P))$.

(*) $[K(X) : K(\mathbb{P}^1)] = 2$, $\exists\, \tilde{\sigma} \in \mathrm{Aut}(K(X)/K(\mathbb{P}^1))$, $\tilde{\sigma}^2 = 1$. By lect 7 $\tilde{\sigma}$ corr to $\sigma \in \mathrm{Aut}\, X$

we classify ell curves ($\cong \Leftrightarrow j = j$)

(Prop) X_1, X_2 ell curves. They are $\cong$ iff $j(X_1)=j(X_2)$. Any value in k arrises as $j(X)$.

Pf. "$\Rightarrow$" from above. "$\Leftarrow$" assume $j(\lambda_1)=j(\lambda_2)$, $\lambda_1 \leftrightarrow X_1$, $\lambda_2 \leftrightarrow X_2$. Now the rat. func $j(t)$ defines a morph $j:\mathbb{P}^1\to\mathbb{P}^1$ of degree 6 [count # of fibres!]. $\#\Sigma = 6$ (+) ; j is inv under Σ so we have $K(\mathbb{P}^1)=K(t)\overset{6}{\supset} K(t)^{\Sigma}\supset K(j(t))$
$\Rightarrow K(j(t)) = K(t)^{\Sigma}$

$\Rightarrow$ the fibres of $j^{-1}(V)\to V$ are precisely the Σ-orbits (3) [bec. if $A=\mathcal{O}(V)$, $B=\mathcal{O}(j^{-1}(V))$ & $m, n\subset B$ ... $m\cap A = n\cap A$ ... $\sigma m \neq n$ $\forall\sigma\in\Sigma$ ... $m\cap A = m\cap A \subset b$ ... $\sigma m \not\subset n$ $\Rightarrow$ $(m\cap A)B \neq b$ .. $\Rightarrow b \neq (b\cap A)B$. Now $\sigma b = b$ $\forall\sigma\in\Sigma$.

$\Rightarrow \lambda_1$ & λ_2 are Σ-conjugated. By verso $X_i \simeq Z(y^2 - x(x-1)(x-\lambda_i))$. But $Z(y^2-x(x-1)(x-\lambda_1)) \overset{(2)}{\simeq} Z(y^2-x(x-1)(x-\lambda_2))$

To check $j(X)$ may take any value j solve $2^8(\lambda^2-\lambda+1)^3 - j\lambda^2(\lambda-1)^2=0$ as eqn in λ ($\lambda=0,1$ are not solutions!).

(Prop) Let $X\subset\mathbb{P}^2$, $X=Z(y^2z - x(x-z)(x-\lambda z))$, $P_0\in X$ the infl. pt $(0:1:0)$ and view X as a group with P_0 = unit (Lect 10). Let $P,Q,R\in X$ (distinct). Then 1) P,Q,R are colin iff $P\oplus Q\oplus R = 0$; 2) P is an inflection pt of X iff P has order 3 on X i.e. $P\oplus P\oplus P = 0$; 3) the line uniting 2 inflection pts of X meets X in a 3^{rd} inflection point.

Pf. 1) P,Q,R colin $\Leftrightarrow \exists$ line L s.t. $X\cdot L = P+Q+R \overset{(4)}{\Leftrightarrow} 3P_0 \sim P+Q+R \Leftrightarrow Cl(P-P_0)+Cl(Q-P_0)+Cl(R-P_0)=0$
2) P inflection pt $\Leftrightarrow \exists$ line L s.t. $X\cdot L = 3P \overset{(4)}{\Leftrightarrow} 3P_0\sim 3P \Leftrightarrow 3Cl(P-P_0)=0$.
3) if P,Q infl pts & $\overline{PQ}\cap X = P+Q+R \overset{1)}{\Rightarrow} P\oplus Q\oplus R = 0 \Rightarrow R = \ominus P\ominus Q$. Since P,Q have ord 3 $\Rightarrow$ so does R!

(Recall) $\operatorname{Aut}\mathbb{P}^1 = PGL(2)$.

(Prop) X elliptic $\Rightarrow \operatorname{Aut} X = T\cdot F$ where $T \simeq$ {gr. of translations on X viewed as gr with P_0 = unit} $\simeq (X,P_0)$, and $F = \{\sigma\in\operatorname{Aut}X ; \sigma(P_0)=P_0\}$, ord $F = \begin{cases} 2 & \text{if } j\neq 0,1728\\ 4 & \text{if } j=1728 \\ 6 & \text{if } j=0\end{cases}$

Pf. Step 1. Analysis of F. Take $f: X\xrightarrow{2}\mathbb{P}^1$, $f(P_0)=\infty$ & $\sigma\in F \Rightarrow \sigma$ induces aut of $|2P_0|$ hence descends to an aut τ of $\mathbb{P}^1$ fixing $\infty \Rightarrow \tau(\{0,1,\lambda\})=\{0,1,\lambda\}$. Now $\tau(t)=at+b \Rightarrow \{b,a+b,a\lambda+b\} = \{0,1,\lambda\}$. Examining all 6 possib $\Rightarrow \lambda\in\{-1,\frac{1}{2},2\}$ ($\Leftrightarrow j=1728$) or $\lambda\in\{-\omega,-\omega^2\}$ ($\Leftrightarrow j=0$; here $\omega^3=1$) & in each case fin many possib for a,b giving τ. There are twice σ's since ~~...~~ if 2 σ's say σ_1,σ_2 give the same τ then $\sigma_1\sigma_2^{-1}$ is the identity on $\mathbb{P}^1 \Rightarrow \sigma_1\sigma_2^{-1} = \langle$ id. ~~interchanging~~ generator of $Gal(K(X)|K(\mathbb{P}^1))$

Step 2. Let $\sigma\in\operatorname{Aut}X$. Let $\tilde\sigma$ be the transl with $P_0\ominus\sigma(P_0)$. Then $(\tilde\sigma\circ\sigma)(P_0) = \tilde\sigma(\sigma(P_0)) = \sigma(P_0)\oplus P_0\ominus\sigma(P_0) = P_0 \Rightarrow \tilde\sigma\circ\sigma\in F$.

(see Lect 12)

(4) "$\Rightarrow$" is clear since $3P_0 = X\cdot L_0$ ($L_0 = Z(x_0)$). "$\Leftarrow$" to check: it is suff to check that the inj map $L(L_0)\to L(3P_0)$ is also surj. But both have dim $= 3$ ($\ell(3P_0)=3$ by RR!)

(3) (L) Let $f: X\to Y$ be mor of non-sing proj curves s.t. $K(Y)\subset K(X)$ is Galois with gr. Σ. Then Σ acts on X, transitively on the fibres.

Pf: If $V\subset Y$ aff, $f^{-1}(V)$ affine. $A=\mathcal{O}(V)$, $B=\mathcal{O}(f^{-1}(V))$. Clearly Σ acts on B (bec B = int. cl of A in $K(X)$) & $B^{\Sigma}=A$ (bec A int. cl). Let $m\subset A$ max & ass $\exists P,Q\subset B$ with $P\cap A = m$, $Q\cap A = m$ but $P\neq\sigma Q$, $\forall\sigma\in\Sigma$.
$\Rightarrow \sigma(P)\neq\tau(Q)$ $\forall\sigma,\tau\in\Sigma \Rightarrow \sigma(P)\not\subset\tau(Q)$ B. Chinese rem. th (Lect 9) $\Rightarrow \exists x\in B$ s.t. $x\equiv 0 \mod \sigma(P)$ $\forall\sigma\in\Sigma$ and $x\not\equiv 0 \mod \tau(Q)$, $\forall\tau\in\Sigma$. Let $y=\prod_{\sigma\in\Sigma}\sigma(x)\in B^{\Sigma}=A$. Since $x\in\sigma^{-1}(P)$, $\forall\sigma\in\Sigma \Rightarrow \sigma(x)\in P \Rightarrow y\in P\cap A = m$. But $x\notin\tau^{-1}(Q)$, $\forall\tau\in\Sigma \Rightarrow \tau(x)\notin Q$, $\forall\tau\in\Sigma \Rightarrow y\notin Q$ / do.

(2) if $\lambda_2 = \frac{1}{\lambda_1}$ use $y' = \lambda^{3/2}y$, $x'=\lambda x$ a.s.o. (examine ?)

(+) $\Sigma = \{t\mapsto t,\ t\mapsto t^{-1},\ t\mapsto 1-t,\ t\mapsto \frac{1}{1-t},\ t\mapsto\frac{t}{t-1},\ t\mapsto\frac{t-1}{t}\}\simeq S_3$ (not Σ_3)

Lecture 14. Hyperelliptic and canonical curves

(For $g \geq 2$ harder)

Up to now we classify completely the non-sing proj curves X with $g = 0, 1$.

Prop Assume X non-sing projective curve of genus $g \geq 2$. Then $|K_X|$ is free and if $\varphi = \varphi_{|K|} : X \to \mathbb{P}^{g-1}$ is the corresponding morphism then 2 sit. may occur:
1) φ is an embedding & $\varphi(X)$ has degree $2g-2$ [X is then called non-hyperelliptic and $\varphi(X) \subset \mathbb{P}^{g-1}$ is called a can. curve]
2) $\varphi(X) \simeq \mathbb{P}^1$, $\varphi(X)$ has degree $g-1$ and $\varphi : X \to \varphi(X)$ has degree 2 [X called hyperell].

Pf. By Lect 12 (γ) for freeness of $|K|$ suff $\dim |K-P| = \dim |K| - 1$, $\forall P \in X$. RR appl to $P \Rightarrow \ell(P) - \ell(K-P) = 1 + 1 - g \Rightarrow \ell(K-P) = \ell(P) + g - 2$. $X \not\simeq \mathbb{P}^1 \Rightarrow \ell(P) = 1 \Rightarrow$ qed// ($\ell(K) = g$). Assume now φ is not an embedding. By Lect 12 $\overset{\text{Lect 10}(\beta)}{\Rightarrow} \exists P, Q \in X$ s.t. $\ell(K-P-Q) > \ell(K) - 2 = g-2$
RR $\Rightarrow \ell(P+Q) - \ell(K-P-Q) = 2 + 1 - g \Leftrightarrow \ell(P+Q) > 1 \Leftrightarrow \dim |P+Q| \geq 1$. Let $R+S \in |P+Q|$
$R+S \neq P+Q$. We claim $\varphi(R) = \varphi(S)$ [if $\varphi(R) \neq \varphi(S)$ $\exists$ hyperpl $H \subset \mathbb{P}^{g-1}$; $R \in H$, $S \notin H$; $\Rightarrow \varphi^* H \in |K|$ contains R but not S in its support $\Rightarrow S \notin Bs|K-R|$
$\Rightarrow \ell(K-R-S) = \ell(K-R) - 1 = \ell(K) - 2$ ↯]. Since $\exists \infty$ $R+S \overset{\text{as above}}{\Rightarrow} \varphi$ not birational. Let
$\varphi : X \to X' = \varphi(X) \subset \mathbb{P}^{g-1}$, $\mu = \deg \varphi \geq 2$, $\tilde{X'} \xrightarrow{\nu} X'$ normalisation, $\tilde{\varphi} : X \to \tilde{X'}$ the induced morph. Define $d = \deg \nu^* H$ ($H \subset \mathbb{P}^{g-1}$ a hyperplane). Lect 9 $\Rightarrow \deg \tilde{\varphi}^*(\nu^* H) =$
$= \mu \cdot d \Rightarrow 2g-2 = \mu d \Rightarrow d \leq g-1$. So we are done by the following (applied to $Y = \tilde{X'}$, $D = \nu^* H$)

L Let Y be non-sing proj curve, $D \in \mathrm{Div}(Y)$, $D > 0$ s.t. $\deg D \leq \dim |D|$. Then $Y \simeq \mathbb{P}^1$ & equality holds. In partic if $\nu : Y \to \mathbb{P}^n$ is a non-deg morph & for $H \subset \mathbb{P}^n$ hyperplane we have $\deg \nu^* H \leq n \Rightarrow$ (eq. holds) $Y \simeq \mathbb{P}^1$ & ν is lin normal $= \nu_{|nP|}$, $P \in Y$ (i.e. ν is the Veronese emb).

Pf. RR for $D \Rightarrow \ell(K-D) \geq g = g(Y)$. If $g \geq 2$ $|K|$ free $\Rightarrow \ell(K-D) \leq \ell(K) - 1 = g-1$ ↯.
If $g = 1 \Rightarrow K \sim 0 \Rightarrow \ell(K-D) = \ell(-D) = 0$ ↯. So $g = 0$. RR for D again $\overset{D>0}{\Rightarrow} \ell(D) = \deg D + 1$
$\Rightarrow \deg D = \dim |D|$ qed//.

(α) Rem. If $\exists$ degree 2 mor $X \to \mathbb{P}^1 \Rightarrow X$ hyperell [Pf: $\exists P, Q \in X$ s.t. $\ell(P+Q) > 1$. By RR $\Rightarrow \ell(K-P-Q) > g-2 = \ell(K) - 2 \underset{\text{as above}}{\Rightarrow} \varphi(P) = \varphi(Q) \Rightarrow \varphi$ not emb!]

Cor Let X_1, X_2 be non-sing proj (non-hyperell) curves of genus $g \geq 2$ and $\sigma : X_1 \to X_2$ an isomorphism. Let $Y_1, Y_2 \subset \mathbb{P}^{g-1}$ their canonical images ($Y_i = \varphi_{|K_{X_i}|}(X_i)$). Then Y_1, Y_2 are proj. equiv.
Pf. σ^* induces a lin. isom $|K_{X_2}| \xrightarrow{\sim} |K_{X_1}|$ which must send $Y_2 \to Y_1$.

NP **L** Let G be a closed subgp of $GL(n)$ of dim > 0. Then G cont. a closed subgroup isom to $\mathbb{G}_m$ or $\mathbb{G}_a$ [char 0 as usual!]. [Pf: one uses Jordan decomp...]

L Let $P_1, \dots, P_n \in \mathbb{P}^1$ distinct pts. Then $\mathrm{Aut}(\mathbb{P}^1; \{P_1, \dots, P_n\}) = \{\sigma \in \mathrm{Aut}\, \mathbb{P}^1 / \sigma(\{P_1, \dots, P_n\}) =$
$= \{P_1, \dots, P_n\}\}$ is finite if $n \geq 3$ [Pf. May ass $P_1 = 0$, $P_2 = 1$, $P_3 = \infty$.
Consider $\mathrm{Aut}(\mathbb{P}^1; \{P_1, \dots, P_n\}) \xrightarrow{\alpha} \Sigma_n$. Suff $\ker \alpha = 1$. If $\sigma \in \ker \alpha$, $\sigma(t) = at + b$
$\sigma(0) = 0$, $\sigma(1) = 1 \Rightarrow b = 0$, $a = 1 \Rightarrow \sigma = \mathrm{id}$]

Cor Let X be non-sing proj curve of genus ≥ 2. Then $\mathrm{Aut}\, X$ is finite.
Pf. Case 1. X is hyperell. Take $\varphi = \varphi_{|K|} : X \to \mathbb{P}^{g-1}$ can morph, $X' = \varphi(X) = \mathbb{P}^1$. Any $\sigma \in \mathrm{Aut}\, X$ induces an aut of $|K|$ so of $\mathbb{P}^1$ preserving the image of ram locus $\{P_1, \dots, P_n\}$ (which has by Hur $n = 2g + 2 \geq 6$). So $\mathrm{Aut}\, X \xrightarrow{\beta} \underbrace{\mathrm{Aut}(\mathbb{P}^1; \{P_1, \dots, P_n\})}_{\text{finite by (L)}}$. Now $\ker \beta$ has card 2 [gen by $\mathrm{Gal}(K(X) : K(\mathbb{P}^1))$

Case 2 - X non-hyperell with can image Y. By (L)! $\mathrm{Aut}\, X = \mathrm{Aut}(\mathbb{P}^g; Y) = \{\bar\sigma \in PGL(g) \mid \bar\sigma Y = Y\} \xleftarrow{\pi} \{\sigma \in SL(g) \mid \bar\sigma Y \subset Y\} = G$ (π finite fibres). Clearly G closed in $GL(g)$
[if $\sigma_0 \in GL(g) \setminus G \Rightarrow \exists y_0 \in Y$ s.t. $\bar\sigma_0 y_0 \notin Y$; take morph $u: GL \to \mathbb{P}^g$, $u(\sigma) = \bar\sigma y_0$; then $\sigma_0 \in u^{-1}(\mathbb{P}^g \setminus Y)$ open $\subset GL(g)\setminus G$] (Ass $\dim G > 0$) By (C)! G contains $G' = \mathbb{G}_a$ or $\mathbb{G}_m$. Take any $y \in Y$ and the mor $u_y^*: G' \to Y$, $u_y^*(\sigma) = \bar\sigma y$. This u_y^* must be const (otherwise induces $\mathbb{P}^1 \to Y$ contradicting Lüroth!!) so any $\sigma \in G'$ acts trivially on Y $\Rightarrow$ ($G' \subset \ker \pi$) $G' = 1$ contradiction. So $\dim G = 0$. Hence G finite. Hence $\mathrm{Aut}\, X$ finite.

(Hur th) Prop ($\exists$ const $\gamma > 0$) X non-sing proj curve of genus $g \geq 2 \Rightarrow \# \mathrm{Aut}\, X \leq$ [illegible]$(g-1)$.

Pf. By prop ! $\#\mathrm{Aut}\, X = n < \infty$; call $\Sigma = \mathrm{Aut}\, X$, $L = K(X)^\Sigma$. To $L \subset K(X)$ there corr morph $f: X \to Y$ of curves. By Lect 13 (2nd page) Σ acts transitively on the fibres $\Rightarrow$ whenever $f(P_1) = f(P_2) \Rightarrow e_{P_1} = e_{P_2}$. By Hur. $(2g-2) = n(2g(Y)-2) + \sum_{i=1}^{s} \frac{n}{r_i}(r_i - 1)$

r_1, r_2 ; $Q_1 \; Q_2 \; Q_s$ [where $Q_1, \dots, Q_s$ = im of ram pts; above each Q_i lie $\frac{n}{r_i}$ pts where $r_i = e_P$ for $f(P) = Q_i$]

so $(2g-2)/n = 2g(Y) - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{r_i}\right)$. Then use:

(L) ($\exists$ const $\gamma > 0$ s.t.) $\forall q \geq 0, s \geq 0, r_i \geq 2$ integers s.t. $2q - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{r_i}\right) > 0$, then $2q - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{r_i}\right) \geq \gamma$

Pf. Ass $\exists$ seq $q_n, s_n, r_{i,n}$ of integers s.t. $\gamma_n = 2q_n - 2 + \sum_{i=1}^{s_n}\left(1 - \frac{1}{r_{i,n}}\right) \downarrow 0$. Extract subseq s.t. each of these is either constant or $\to \infty$ (strictly $\nearrow$). Now $\gamma_{n+1} - \gamma_n = 2\underbrace{(q_{n+1} - q_n)}_{\geq 0} + \underbrace{s_{n+1} - s_n}_{\geq 0} + \sum_{i=1}^{s_n}\left(\frac{1}{r_{i,n}} - \frac{1}{r_{i,n+1}}\right) - \frac{1}{r}$

If s_n constant $\Rightarrow \gamma_{n+1} - \gamma_n \geq 0 \Rightarrow \gamma_n \nearrow$ contradiction.

If $s_n \uparrow \infty$ strictly $\Rightarrow \gamma_{n+1} - \gamma_n \geq 1 - \frac{1}{r_{s_{n+1}, n+1}} > 0 \Rightarrow \gamma_n \uparrow 0 \Rightarrow$ contradiction.

Rem One can show $\gamma = 84$!

Prop X non-sing proj curve of genus $2 \Rightarrow X$ hyperelliptic.

Pf. $\varphi_{|K|}: X \to \mathbb{P}^{g-1} = \mathbb{P}^1$ cannot be an immersion!

Prop X non-sing proj non-hyperell curve of genus $3 \Leftrightarrow X \simeq$ plane quartic

Pf "$\Rightarrow$" $\varphi_{|K|}: X \to \mathbb{P}^{g-1} = \mathbb{P}^2$ emb $\Rightarrow \deg(\varphi(X)) = \deg K = 2g - 2 = 4$

"$\Leftarrow$" X plane quartic $\Rightarrow g(X) = \frac{(d-1)(d-2)}{2} \underset{d=4}{=} 3$ & $K_X \sim (d-3)L = L$ (cf. Lect..) & (if $i: X \hookrightarrow \mathbb{P}^2$ is the inclusion) $i^*|H|$ ($\subset |K_X|$) has dimension 2 (same as $|H|$). But $\dim |K_X| = g - 1 = 2$. So $i^*|H|$ is complete so $i = \varphi_{|K|}$ so $\varphi_{|K|}$ embedding!

NP Rem $\forall g \geq 3$ $\exists$ non-hypell curves of genus g [we proved it only for $g = 3$! Similarly for $\forall g$ of the form $\frac{(d-1)(d-2)}{2}$ (look at plane curves!)]

Rem $\forall g \geq 2$ $\exists$ hyperell curves of genus g [Pf: Let $\varphi_{2g+1} \in k[x]$ with dist roots, $X_0 = Z(y^2 - \varphi(x)) \subset \mathbb{A}^2$, X = non-sing model. Clearly $\exists$ mor $f: X \to \mathbb{P}^1$ of deg 2. So by (*) suff to check $g(X) = g$. This foll by Hur: f is ramif at $2g+2$ pts (the $2g+1$ roots of φ and ∞; ∞ appears since if it would miss we would get in Hur formula a contrad mod 2)

(*) Y nondeg $\Rightarrow \exists$ g proj indep pts in $\mathbb{P}^{g-1}$ on Y (i.e. not cont in a hyp). We may ass they are $(1:0\cdots0), \dots, (0:0\cdots0:1)$. A proj transf fixing these pts is the identity.

Lecture 15. Proof of RR, I: sheaves, cohomology.

Def X a top. space. A presheaf $\mathcal F$ of ab. gr. is an assoc. U (open) $\mapsto \mathcal F(U)$ (ab. gr.) and ab. gr. homo $\rho_{UV}: \mathcal F(U)\to \mathcal F(V)$ (for $\forall\, V\subset U$) s.t. 1) $\mathcal F(\emptyset)=0$, 2) $\rho_{UU}=id$, 3) $\rho_{UW}=\rho_{VW}\circ\rho_{UV}$ ($\forall\, W\subset V\subset U$). Similar def. for "vect. sp.", "rings" a.s.o. Call $s|_V=\rho_{UV}(s)$. Write $H^0(U,\mathcal F)=\mathcal F(U)$.

Def $\mathcal F$ called a sheaf if $\forall\, U=\cup U_i$ open cov. we have 4) if $s\in\mathcal F(U)$, $s|_{U_i}=0$ $\forall i \Rightarrow s=0$ and 5) if $s_i\in\mathcal F(U_i)$ are s.t. $s_i|_{U_i\cap U_j}=s_j|_{U_i\cap U_j} \Rightarrow \exists\, s\in\mathcal F(U)$ s.t. $s|_{U_i}=s_i$ $\forall i$. Call $Ab(X)$ ...

Ex X var $\Rightarrow (U\mapsto \mathcal O(U))$ sheaf $\mathcal O_X$. If X curve non-sing $(U\mapsto \Omega(U))$ sheaf Ω_X. (*)

Def X top. sp. $\mathcal F$ presheaf. $\mathcal F$ called const. if ...

Ex Const. presheaf on X irred. is a sheaf.

Def A morph. of presheaves is ... So $Ab(X)$ becomes a categ.

Def If $\mathcal F$ presheaf, $P\in X$ define stalk $\mathcal F_P := \varinjlim_{P\in U}\mathcal F(U)$ (2). Any mor $\varphi:\mathcal F\to\mathcal G$ induces $\varphi_P:\mathcal F_P\to\mathcal G_P$, $\forall P$.

(α) **Prop** If $\varphi:\mathcal F\to\mathcal G$ mor in $Ab(X)$ then φ iso iff $\varphi_P:\mathcal F_P\to\mathcal G_P$ for all $P\in X$. (not true for presh.)

Pf. Ass. φ_P iso $\forall P$. Want $\varphi(U):\mathcal F(U)\to\mathcal G(U)$ iso $\forall U$. Inj: take $s\in\mathcal F(U)$, $s\mapsto 0\in\mathcal G(U) \Rightarrow s_P\mapsto 0\in\mathcal G_P$ $\forall P \Rightarrow s\mapsto 0\in\mathcal F_P$ $\forall P$ $\left[\begin{matrix}\mathcal F(U)\to\mathcal G(U)\\ \downarrow \quad\quad \downarrow\\ \mathcal F_P\to\mathcal G_P\end{matrix}\right] \Rightarrow s=0$. Surj. analogously (lift locally then glue).

(β) **Prop** $\forall$ presheaf $\mathcal F$ $\exists$ sheaf $\mathcal F^+$ & mor. of presh. $\mathcal F\to\mathcal F^+$ s.t. $\forall\, \mathcal F\to\mathcal G$ sheaf $\Rightarrow \exists!\ \mathcal F^+\to\mathcal G$.

Pf: $\mathcal F^+(U):=\{$functions $s: U\to\coprod_{P\in U}\mathcal F_P$ | $\forall P\in U$ $s(P)\in\mathcal F_P$ & $\exists$ nbhd V of P in U and $t\in\mathcal F(V)$ s.t. $\forall Q\in V$ $s(Q)=t_Q\}$ (4). Then check. **Rem** $\mathcal F$ and $\mathcal F^+$ have the same stalks! **Rem** $\mathcal F$ sheaf $\Rightarrow \mathcal F^+=\mathcal F$.

Def If $\mathcal F'$ subsheaf of $\mathcal F$ (i.e. $\mathcal F'(U)\subset\mathcal F(U)$). Define sheaf $\mathcal F/\mathcal F' = \left(U\mapsto \frac{\mathcal F(U)}{\mathcal F'(U)}\right)^+$. Note (3) $(\mathcal F/\mathcal F')_P=\frac{\mathcal F_P}{\mathcal F'_P}$

Def A seq. $\mathcal F'\xrightarrow{\alpha}\mathcal F\xrightarrow{\beta}\mathcal F''$ is called exact if $\mathcal F'_P\to\mathcal F_P\to\mathcal F''_P$ ex. $\forall P$ [$\not\Rightarrow \mathcal F'(U)\to\mathcal F(U)\to\mathcal F''(U)$ ex. $\forall U$!!!]

Rem If $\mathcal F'\subset\mathcal F$ subsheaf $\Rightarrow 0\to\mathcal F'\to\mathcal F\to\mathcal F/\mathcal F'\to 0$ exact by

Rem $0\to\mathcal F'\to\mathcal F\to\mathcal F''\to 0$ exact $\Rightarrow 0\to H^0(U,\mathcal F')\to H^0(U,\mathcal F)\to H^0(U,\mathcal F'')$ and (6). [Pf. May ass. $U=X$: Ex. in I checked as in (α), Ex. in II: let $s\in H^0(X,\mathcal F)$, $s\mapsto 0\in H^0(X,\mathcal F'') \Rightarrow s\mapsto 0\in\mathcal F''_P$, $\forall P\in X \Rightarrow s_P=s'(P)_P$ where $s'(P)\in\mathcal F'_P \Rightarrow$ can find covering V_i of X and $s'_i\in\mathcal F'(V_i)$ s.t. $s_P=(s'_i)_P$ $\forall P\in V_i \Rightarrow (s'_i)_P=(s'_j)_P$ $\forall P\in V_i\cap V_j$ $\Rightarrow s'_i|_{V_i\cap V_j}=s'_j|_{V_i\cap V_j}$ (5) $\Rightarrow s'_i$ glue together by axiom 5) to give s' qed.]

Def X top. sp., $P\in X$ closed pt. A an abel. gr. Define the skyscraper sheaf $A_P\in Ab(X)$ by $A_P(U)=A$ if $U\ni P$ and $A_P(U)=0$ if $U\not\ni P$. Its stalks are A at P and 0 in any $Q\neq P$.

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(5) If $\mathcal F$ sheaf and $s\in\mathcal F(U)$, $s_P=0$ $\forall P\in U \Rightarrow s=0$ (by axiom 4))

(4) Alternatively an element of $\mathcal F^+(U)$ is a collection (V_i,t_i) where $\cup V_i=U$, $t_i\in\mathcal F(V_i)$ s.t. $(t_i)_P=(t_j)_P$ $\forall P\in V_i\cap V_j$ modulo the rel. $(V_i,t_i)\sim(V'_j,t'_j)$ iff $(t_i)_P=(t'_j)_P$ $\forall P\in V_i\cap V'_j$.

(3) bec. lim commute with quot. $(\mathcal F/\mathcal F')_P=\varinjlim_{P\in U}\frac{\mathcal F(U)}{\mathcal F'(U)}=\frac{\varinjlim\mathcal F(U)}{\varinjlim\mathcal F'(U)}=\frac{\mathcal F_P}{\mathcal F'_P}$

(2) If X curve $\mathcal O_{X,P}$ as stalk is nothing but loc. ring defined... Same with $\Omega_{X,P}$ (by very def).

Direct def. $\mathcal F_P=\{(U,s);\ s\in\mathcal F(U), P\in U\}/\sim$.

(*) if $D\in Div\,X$, then define sheaves $\mathcal O(D)$, $\Omega(D)$ by $\mathcal O(D)(U)=\{f\in K(X)^*\mid v_P(f)+v_P(D)\geq 0,\ \forall P\in U\}\cup\{0\}$, $\Omega(D)(U)=\{\omega\in\Omega_{K(X)/k}\mid v_P(\omega)+v_P(D)\geq 0\ \ \forall P\in U\}$. So $H^0(X,\mathcal O(D))=L(D)$, $H^0(X,\Omega(D))\simeq L(K_X+D)$; the latter is given by $f\omega\leftarrow f$ if $K_X=(\omega)$, $\omega\in\Omega_{K(X)/k}$.

(6) $\mathcal F''\simeq\mathcal F/\mathcal F'$ (bec. $\exists$ nat. morph. $\mathcal F/\mathcal F'\to\mathcal F''$ by (β) which is iso by (α)

Def X top space, $\underline{\mathcal{U}} = (U)$ a covering. Define $Z^1(\mathcal{U}, \mathcal{F}) = \{(a_{UV}) \in \prod_{U,V \in \underline{\mathcal{U}}} \mathcal{F}(U \cap V) \mid$ ($\mathcal{U}$ should be viewed as index set; same U's may occur)

$a_{UU} = 0$, $a_{UV} + a_{VW} + a_{WU} = 0$ in $\mathcal{F}(U \cap V \cap W)\}$, $B^1(\underline{\mathcal{U}}, \mathcal{F}) = \{(f_U - f_V) \in \prod_{U,V \in \underline{\mathcal{U}}} \mathcal{F}(U \cap V) \mid f_U \in \mathcal{F}(U)\}$

$H^1(\underline{\mathcal{U}}, \mathcal{F}) = Z^1(\underline{\mathcal{U}}, \mathcal{F}) / B^1(\underline{\mathcal{U}}, \mathcal{F})$. A refinement of $\underline{\mathcal{U}}$ is another covering $\underline{\mathcal{U}}'$ and a map $\underline{\mathcal{U}}' \xrightarrow{\alpha} \underline{\mathcal{U}}$, $V' \mapsto \alpha(V') = V$ s.t. $V' \subset V$. $\exists$ nat homo $H^1(\underline{\mathcal{U}}, \mathcal{F}) \to H^1(\underline{\mathcal{U}}', \mathcal{F})$

Set of cov. has a partial order. Any 2 cov have a cov which is ref of both.

[an elt repr by (a_{UV}) goes to $(a_{U'V'})$ where $a_{U'V'} := a_{\alpha(U')\alpha(V')}$]. Put $H^1(X, \mathcal{F}) = \varinjlim_{\underline{\mathcal{U}}} H^1(\underline{\mathcal{U}}, \mathcal{F})$

$H^1(X, \cdot)$ is a functor! ($\circledast$)

(8) **Prop** If X top sp, $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \to 0$ ex in $Ab(X)$ $\Rightarrow$ $\exists$ nat ∂ s.t. $0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \xrightarrow{\partial} H^1(X, \mathcal{F}') \xrightarrow{\alpha^1} H^1(X, \mathcal{F}) \xrightarrow{\beta^1} H^1(X, \mathcal{F}'')$. exact (i.e. ker = im.)

Pf. Since $\varinjlim$ (ex seq) = ex seq to check ex in $H^1(X, \mathcal{F})$ it is suff to check for $H^1(\underline{\mathcal{U}}, \cdot)$. This is triv.

Define ∂. Let $x \in H^0(X, \mathcal{F}'') = H^0(X, \mathcal{F}/\mathcal{F}')$, view x as repres by a family $(V, \bar{t}_V)$ where $t_V \in \mathcal{F}(V)$ and $\bar{t}_V = t_V \bmod \mathcal{F}'(V)$. Then the family $(t_U - t_V)$, $t_U - t_V \in \mathcal{F}(U \cap V)$ has the prop $\overline{t_U - t_V} = 0 \Rightarrow t_U - t_V \in \mathcal{F}'(U \cap V)$ so defines an elt in $H^1(X, \mathcal{F}')$, call it $\partial(x)$.

E. Exactness in $H^0(X, \mathcal{F}'')$ and $H^1(X, \mathcal{F}')$ triv checking!

Prop $\mathcal{F}$ flasque (either const sh on X irr or skyscraper) $\Rightarrow H^1(X, \mathcal{F}) = 0$. ~~In partic this holds for skyscrapers & for const sh…~~ (s.t. $P \in W$ if $\mathcal{F} = A_P$)

Pf. Let $x \in H^1(X, \mathcal{F})$ be repres by (a_{UV}) for some cov $\underline{\mathcal{U}}$. Fix $W \in \underline{\mathcal{U}}$ $\Rightarrow$ $\forall U, V \in \underline{\mathcal{U}}$ we have $a_{UV} = a_{WV} - a_{WU}$ in $U \cap V \cap W$. Lift each $a_{WD} \in \mathcal{F}(W \cap D)$ to some $a_D \in \mathcal{F}(D)$ ($\forall D \in \underline{\mathcal{U}}$). Then $a_{UV} = a_V - a_U$. Since $\mathcal{F}(U \cap V) \to \mathcal{F}(U \cap V \cap W)$ is $\circ$ $\Rightarrow a_{UV} = a_V - a_U$ on both $U \cap V$ qed.

Def (3) If $\mathcal{F}$ sheaf of k-lin spaces $\Rightarrow H^i(X, \mathcal{F})$ $i = 0, 1$ are k-lin sp. If of fin dim def $\chi(\mathcal{F}) = \dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F})$ (we say $\chi(\mathcal{F})$ is defined and) (Euler char).

Prop If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ ex seq of sheaves of k-lin sp s.t. all 3 Euler char are defined (s.t. $H^1(\mathcal{F}'') = 0$). Then

E $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. [Pf: cf (8) + general fact: if $0 \to V_1 \to V_2 \to \cdots \to V_m \to 0$ exact of vect sp i.e. $\mathrm{Im}\, \alpha_i = \mathrm{Ker}\, \alpha_{i+1}$ then $\sum (-1)^i \dim V_i = 0$ if all $\dim V_i < \infty$]

Prop Let X be a non-sing proj curve. Then for all $D \in \mathrm{Div}(X)$, $\chi(\mathcal{O}(D))$ is defined and $\chi(\mathcal{O}(D)) = \deg D + \chi(\mathcal{O})$.

Pf. Suff to check that ↑ holds for D iff it holds for $D + P$ ($P \in X$) bec. clearly it holds for $D = 0$. Let $D = \sum n_Q Q$. We have ex seq $0 \to \mathcal{O}(D) \to \mathcal{O}(D+P) \to k_P \to 0$ given by $0 \to \mathcal{O}(D)(U) =$

$= \{f \mid v_Q(f) + n_Q \geq 0 \ \forall Q \in U\} \to \mathcal{O}(D+P)(U) = \{f \mid v_Q(f) + n_Q + \delta_{PQ} \geq 0 \ \forall Q \in U\} \to \left(\frac{t_P^{-n_P - 1}\mathcal{O}_P}{t_P^{-n_P}\mathcal{O}_P}\right)_P (U) \to 0$, $t_P =$ = par of $\mathcal{O}_P$ (Kronecker) $\overset{(*)}{\Rightarrow}$ (2) $\chi(\mathcal{O}(D+P)) = \chi(\mathcal{O}(D)) + 1$

$k_P(U) = \begin{cases} 0 & \text{if } P \notin U \\ k & \text{if } P \in U \end{cases}$

Cor To prove RR it is suff to prove that $\dim H^1(\mathcal{O}(D)) = \dim H^0(\mathcal{O}(K - D))$. [Pf: …]

(*) it is exact bec. $\mathcal{O}(D)_P = \{f \mid v_P(f) + n_P \geq 0\} = t^{-n_P}\mathcal{O}_P$ & simil. for $\mathcal{O}(D+P)$

(2) We already know (Lect 12) that the H^0's are fin dim. So $H^1(\mathcal{O}(D))$ fin dim iff $H^1(\mathcal{O}(D+P))$ is so!

(3) This def of Euler char is good only for dim 1. Should be called "cripple Euler char".

Finite dimensionality of $H^1(X, \mathcal{O})$ to be proved separately

Lecture 16. Proof of RR, II: residues

(Def) Let k field of ch. 0, $\mathcal{O}$ a DVR, $k\subset \mathcal{O}$, $m\subset\mathcal{O}$ max, $\mathcal{O}/m\simeq k$. Define its completion $\hat{\mathcal{O}} := \varprojlim \mathcal{O}/m^n = \{(a_0,a_1,a_2,\dots)\in \frac{\mathcal{O}}{m}\times\frac{\mathcal{O}}{m^2}\times\frac{\mathcal{O}}{m^3}\times\dots \mid a_{i+1} \bmod m^i = a_i \ \forall i\}$. It is a ring & $\mathcal{O}\hookrightarrow\hat{\mathcal{O}}$.

(Prop) Any par $t\in m$ defines an iso $\hat{\mathcal{O}}\simeq k[[T]]$, $t\mapsto T$. (in partic $\hat{\mathcal{O}}$ is a DVR too!)

Pf. $\exists\, k[[T]]/(T^n)\to\mathcal{O}/(t^n)$, $T\mapsto t$ clearly inj. By Lect 3 same dim $\Rightarrow$ bij. Pass to $\varprojlim$.

(Def) $\mathcal{O}$ called complete if $\mathcal{O}=\hat{\mathcal{O}}$. (Convention) We write $\uparrow$ $k[[t]]$ instead of $k[[T]]$.

(Rem) $Q(\mathcal{O})\simeq k((t)) = \{\sum_{i\gg-\infty} a_i t^i \mid a_i\in k\}$ (= Laurent series).

(Def) Ass $\mathcal{O}$ complete. Let $K=Q(\mathcal{O})$, $M\subset\Omega_{K/k}$, $M=\bigcap_{n\geq 1}\mathcal{O}t^n dt$ (does not depend)

$\Omega'_{K/k}=\Omega_{K/k}/M$. , $d': K\xrightarrow{d}\Omega_{K/k}\longrightarrow\Omega'_{K/k}$.

(×) (Prop) In hyp $\uparrow$. 1) $\Omega'_{K/k}$ is a K-lin sp of dim 1; if $t\in\mathcal{O}$ is any param $d't$ is a basis.
2) if $f=\sum_{n\gg-\infty}\lambda_n t^n\in K$ then $d'f=\frac{\partial f}{\partial t}d't$ where $\frac{\partial f}{\partial t}=\sum n\lambda_n t^{n-1}$.

Pf: 2) Check $df-\frac{\partial f}{\partial t}dt\in\mathcal{O}t^N dt$ $\forall N$. Write $f=f_0+t^{N+1}f_1$, $f_0=\sum_{n\leq N}\lambda_n t^n$, $f_1\in k[[t]]$
$\Rightarrow \frac{\partial f}{\partial t}=\frac{\partial f_0}{\partial t}+t^N g$, $g\in k[[t]]$ $\Rightarrow df-\frac{\partial f}{\partial t}dt=(N+1)t^N f_1+t^{N+1}df_1-t^N g\,dt\in k[[t]]t^N dt$.
1) By 2) $d't$ generates $\Omega'_{K/k}$; it is suff to check $\Omega'_{K/k}\neq 0$. Ass $\Omega'_{K/k}=0$ i.e. $M=\Omega_{K/k}$. Let $\frac{\partial}{\partial t}:k((t))\to k((t))$ deriv. By univ $\frac{\partial}{\partial t}$ induces $\Omega_{K/k}\xrightarrow{\alpha}K$; α vanishes on M [bec if $h\in k[[t]]$, $\alpha(ht^n dt)=ht^n\in m^n$ so $\alpha(M)\subset\bigcap m^n=0$] i.e. α vanishes on $\Omega_{K/k}$ $\Rightarrow\frac{\partial}{\partial t}=0$ $\not\Diamond$. OK. (Convention) Write d instead of d'!

(Def) Let $\mathcal{O}$ be complete, $t\in\mathcal{O}$ par, $K=Q(\mathcal{O})$, $\omega\in\Omega'_{K/k}$. Define $\mathrm{Res}_t(\omega)=a_{-1}\in k$, where $\omega=f\,dt$, $f\in k((t))$, $f=\sum_{i\gg-\infty}a_i t^i$.

(α) (Prop) 1) $\omega\mapsto\mathrm{Res}_t(\omega)$ is k-lin, 2) $\mathrm{Res}_t(\omega)=0$ if $\omega\in\mathcal{O}dt$ (*), 3) $\mathrm{Res}_t(df)=0$ for $f\in K$
4) $\mathrm{Res}(dg/g)=v(g)$ (where v the val on K) $\forall g\in K^*$. [Pf: clear].

(Prop) $\mathcal{O}$ complete, $K=Q(\mathcal{O})$, $t,s\in\mathcal{O}$ 2 par, $\omega\in\Omega'_{K/k}$. Then $\mathrm{Res}_t(\omega)=\mathrm{Res}_s(\omega)$ (2)

Pf. Write $\omega=\sum_{n\geq 0}a_n\frac{ds}{s^n}+\omega_0$, $v(\omega_0)\geq 0\Rightarrow\mathrm{Res}_s(\omega)=a_1$. Now $\mathrm{Res}_t(\omega)\overset{(\alpha)}{=}$
$=\sum a_n\mathrm{Res}_t\left(\frac{ds}{s^n}\right)$. Now $\mathrm{Res}_t\left(\frac{ds}{s}\right)\overset{(\alpha)}{=}v(s)=1$ and $\underbrace{\mathrm{Res}_t\left(\frac{ds}{s^n}\right)}_{\text{for } n\geq 2}=\mathrm{Res}_t\left(d\left[-\frac{1}{(n-1)s^{n-1}}\right]\right)\overset{(\alpha)}{=}0$

(Def) If $\mathcal{O}$ is not nec. complete, $E=Q(\mathcal{O})$, $K=Q(\hat{\mathcal{O}})$, $\omega\in\Omega_{E/k}$ define $\mathrm{Res}(\omega)=$
$=\mathrm{Res}(\hat{\omega})$ where $\hat{\omega}$ = image of ω via $\Omega_{E/k}\to\Omega_{K/k}\to\Omega'_{K/k}$. In partic if X non-sing curve, $P\in X$ & $\omega\in\Omega_{K(X)/k}$ we have defined $\mathrm{Res}_P(\omega)\in k$.

(Th) (Residue formula) If X non-sing proj curve, $\omega\in\Omega_{K(X)/k}\Rightarrow\sum_{P\in X}\mathrm{Res}_P(\omega)=0$.
We prove it in several steps.

(*) This cond is indep of the choice of t (we write $v(\omega)\geq 0$ for it) | (2) So we may define $\mathrm{Res}(\omega)$.

(L1) Res. formula holds for $\mathbb{P}^1$

Pf. Any $f \in k(t)$ is sum of fns λt^n & $\mu \frac{1}{(t-a)^n}$ So we may ass 1) $\omega = t^n dt$, 2) $\omega = \frac{dt}{(t-a)^n}$

If 1) the only pole is ∞; put $s = \frac{1}{t} \Rightarrow \omega = -s^{-n-2}ds \Rightarrow Res_\infty(\omega) = 0$. If 2) with $n \geq 2$ a single pole $t=a$ with res 0; if $n=1$ 2 poles $t=a$, $t=\infty$ with residues $+1$ & -1.

(Constr) Let $f: X \to X' = \mathbb{P}^1$ be non-const morph, $F = k(X) \supset E = k(X')$. We dispose of $Tr = Tr_{F/E}: F \to E$ (*). By Lect 11 $\forall$ elt in $\Omega_{F/k}$ has the form $f dg$ ($f \in F, g \in E$).

Define $Tr_{F/E}: \Omega_{F/k} \to \Omega_{E/k}$, $Tr(f dg) = Tr(f) dg$ [well def: if $f dg = f_1 dg_1$, $f_1 \in F$, $g_1 \in E$, $dg_1 = h dg$, $h \in E \Rightarrow Tr(f_1) dg_1 = Tr(\frac{f}{h}) h dg = Tr(f) dg$]. By (L1), (Th) foll from

(L2) For any $Q \in X'$, $\sum_{f(P)=Q} Res_P(\omega) = Res_Q(Tr(\omega))$.

To prove (L2) write $E_Q = Q(\hat{\mathcal{O}}_Q)$, $F_P = Q(\hat{\mathcal{O}}_P)$. We have comm diagr
$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ E_Q & \longrightarrow & \prod_{P \mapsto Q} F_P \end{array}$$

One can show $\forall$ basis of F as E-lin sp is a basis of $\prod F_P$ as E_Q-lin space (2)

$\Rightarrow \forall f \in F$, $Tr_{F/E}(f) = \sum_{P \mapsto Q} Tr_{F_P/E_Q}(f) \Rightarrow Res_Q(Tr_{F/E}(f) dg) = \sum_{P \mapsto Q} Res_Q(Tr_{F_P/E_Q}(f) dg)$ so it is suff to check

(L3) $Res_Q(Tr_{F_P/E_Q}(f) dg) = Res_P(f dg)$ for $\forall f \in F_P$, $\forall g \in E_Q$

see Bourbaki

Pf. By (2) $E_Q = k((t))$, $F_P = k((s))$ where $t = s^e$. Now (3)

So we must check $Res_t(Tr_{k((s))/k((t))}(s^n) dt) = Res_s(s^n dt)$. Let $1, s, \dots, s^{e-1}$ be the basis of $k((s))/k((t))$. We have $Res(s^n dt) = Res(s^n d(s^e)) = Res(e s^{n+e-1} ds) = \begin{cases} 0 & \text{if } n \neq -e \\ e & \text{if } n = -e. \end{cases}$

On the other hand the matrix of ϕ_{s^n} (= mult by s^n) w.r.t. the basis $1, \dots, s^{e-1}$ is: $\begin{pmatrix} t^m & & \\ & t^m & \\ & & \ddots \\ & & & t^m \end{pmatrix}$ if $n = me$ and $\begin{pmatrix} 0 & \cdots & t^m & & \\ & & & \ddots & \\ & & & & t^m \\ t^{m+1} & & & & \\ & \ddots & & & \\ & & t^{m+1} & & 0 \end{pmatrix}$ if $n = me + k$, $1 \leq k \leq e-1$.

So $Tr(s^n) = \begin{cases} 0 & \text{if } n \not\equiv 0 (e) \\ e t^{\frac{n}{e}} & \text{if } n \equiv 0 (e) \end{cases}$. So $Res_t(Tr(s^n) dt) = \begin{cases} 0 & \text{if } n \neq -e \\ e & \text{if } n = -e \end{cases}$ qed//

(3) It is suff to check (L2) for $g = t$ ⟶ (bec by (+) $\forall g \in E$ we have $dg = h dt$ in Ω' !), $h \in E$
and it is suff to ass $f = \sum_{n \geq n_0} a_n s^n$ (bec $Tr(k[[s]]) \subset k[[t]]$) so it is suff to ass $f = s^n$.

(2) At least dimensions are the same. Indeed

Each $[F_P : E_Q] = e_P$ (ramif index) because $F_P = k((t_P))$, $E_Q = k((t_Q))$, $t_Q = u t_P^{e_P}$, $u \in k[[t_P]]^*$
Put $v = u^{1/e_P} = \exp(\frac{1}{e_P} \log u)$, $s_P = u^{1/e_P} t_P$ so $t_Q = s_P^{e_P} \Rightarrow k[[t_P]] = k[[s_P]]$ is a finite $k[[t_Q]]$-module, free of rank e_P (gen by $1, s_P, s_P^2, \dots, s_P^{e_P-1}$) hence $k((t_P))$ is a $k((t_Q))$-vect sp. of dim e_P!

(*) $Tr(x) = Tr(\phi_x)$; $\phi_x \in End_E(F)$, $\phi_x(y) = xy$. $x \mapsto Tr\, x$ is E-linear and $Tr|_E = [F:E] \cdot id_E$.

Curves 17. Proof of RR III; adèles and duality

Here k of char 0, X non-sing proj curve.

Def The ring R of adèles is the ring whose elts are families $r = \{r_P\}_{P\in X}$, $r_P \in K(X)$ s.t. $r_P \in \mathcal{O}_P$ for almost all P. If $D \in \mathrm{Div}(X)$ put $R(D) = \{r \in R \mid v_P(r_P) + v_P(D) \geq 0 \text{ all } P\}$.

Clearly $R = \bigcup R(D)$. $\exists$ inj $K(X) \to R$, $f \mapsto r$ where $r_P = f$, $\forall P$.

Prop If $D \in \mathrm{Div}(X)$ then $H^1(X, \mathcal{O}(D)) \simeq R/(R(D) + K(X))$.

Pf. Ex seq. $0 \to \mathcal{O}(D) \to K(X) \to K(X)/\mathcal{O}(D) \to 0 \Rightarrow K(X) = H^0(X, K(X)) \to H^0(X, K(X)/\mathcal{O}(D)) \to H^1(X, \mathcal{O}(D)) \to 0$ (bec $K(X)$ ~~flasque~~ constant) so suff $H^0(X, K(X)/\mathcal{O}(D)) \xrightarrow[\sim]{\alpha} R/R(D)$. But define α as foll. An elt $*$ in H^0 is given (by Lect 15) by a family $(U, \bar{r}_U)$ where (U) finite covering of X, $r_U \in K(X)$, $\bar{r}_U \in K(X)/\mathcal{O}_D$, and $(\bar{r}_U)_P = (\bar{r}_V)_P$ for all U, V, $P \in U \cap V$ i.e. $r_U - r_V \in t\mathcal{O}_P^{-n_P}$ $\forall P \in U \cap V$ ($n_P = v_P(D)$). Define an adèle r by fixing for each P an $U_P \ni P$ and putting $r_P = r_{U_P}$ for $P \in X$.(*)

~~Then put $\alpha(*) = r$ mod $R(D)$. Now for constructing α^{-1} fix $r \in R$; for each P, $\exists\, U_P \ni P$ s.t. r_P has no zeros & poles on $U_P \setminus \{P\}$. Consider the family $(U_P, \bar{r}_P)$ where $\bar{r}_P$ = class of r_P in $K(X)/\mathcal{O}(D)$. For any Q we have $(\bar{r}_P)_S = (\bar{r}_Q)_S$ $\forall S \in U_P \cap U_Q$ (bec $r_P - r_Q$~~

Then put $\alpha(*) = r \bmod R(D)$. Now construct α^{-1}. Fix $r \in R$. Let $D = \sum_{i=1}^{n} n_i P_i$, $r_i := r_{P_i}$. Choose open sets $U_i \ni P_i$ s.t. $U_i \not\ni P_j$ for $j \neq i$ & r_i has no zero or poles in $U_i \setminus \{P_i\}$. Then $X \setminus (U_1 \cup \dots \cup U_n) = \{Q_1, \dots, Q_m\}$. Let $U_i' \ni Q_i$ s.t. $U_i' \cap \{P_1, \dots, P_n, Q_1, \dots, Q_m\} = \{P_i\}$ & $r_j' := r_{Q_j}$ has no poles & zer on $U_j' \setminus \{Q_j\}$. Then $(U_1, \bar{r}_1), \dots, (U_n, \bar{r}_n), (U_1', \bar{r}_1'), \dots, (U_m', \bar{r}_m')$ define an elt $*$ in $H^0(K(X)/\mathcal{O}(D))$ (bec. $r_i - r_j$, $r_i' - r_j$, $r_i' - r_j'$ are regular on $U_i \cap U_j$, $U_i \cap U_j'$, $U_i' \cap U_j'$) & this elt is 0 if $r \in R(D)$ [suf to check $*_P = 0$ $\forall P$ which is clear].

Def Let $J(D) = (R/(R(D) + K(X)))^{\circ}$ (2). For $D' \geq D \Rightarrow R(D') \supset R(D) \Rightarrow J(D) \supset J(D')$.

Put $J = \bigcup J(D)$ (3). Note J is a $K(X)$-linear space [if $f \in K(X)$ and $\alpha \in J$, $\alpha : R \to k$, α vanishing on $K(X) + R(D)$ some. Then define $f \cdot \alpha : R \to k$ by $(f\alpha)(r) = \alpha(fr)$. If $f \in L(D')$ then $f\alpha$ clearly vanishes on $K(X)$ and on $R(D - D')$ so we have $\alpha f \in J(D - D')$ $\big(L(D') R(D-D') \subset R(D)\big)$. Note so $L(D') J(D) \subset J(D - D')$].

(α) **Prop** $\dim_{K(X)} J \leq 1$.

Pf. Ass. $\exists\, \alpha, \alpha' \in J$ lin ind / $K(X)$, Ass $\alpha, \alpha' \in J(D)$, $d = \deg D$. Take any $P \in X$. Then the map $L(nP) \oplus L(nP) \to J(D - nP)$, $(f, g) \mapsto f\alpha + g\alpha'$ is injective so $\dim_k J(D - nP) \geq 2 \dim_k L(nP)$. We will make $n \to \infty$ and get $\not\!\!\!\times$. We have $\dim J(D - nP) \overset{\text{fin dim}}{=} \dim R/(R(D - nP) + K(X)) = \dim H^1(X, \mathcal{O}(D - nP)) = \ell(D - nP) - \chi(\mathcal{O}(D - nP)) \overset{\text{Lect 15}}{=} \ell(D - nP) - \deg(D - nP) - \chi(\mathcal{O}) = \ell(D - nP) + n - (d + \chi(\mathcal{O})) \underset{\text{for } n > d}{=} n - (d + \chi(\mathcal{O}))$.

(3) so $J = \{$lin maps $: R \to k$ which vanish on $K(X)$ and on some $R(D)\}$. The $\bigcup J(D)$ is filtered.

(2) so $J(D) = \{$lin maps $: R \to k$ which vanish on $R(D) + K(X)\}$.

(*) since we have fin many U's the r_U's will have fin many poles so r is indeed an adèle.

On the other hand $\dim L(nP) = \ell(nP) = H^0(X,\mathcal{O}(nP)) \geq \underbrace{\deg(nP)}_{n} + \chi(\mathcal{O})$. We get

$n - (d+\chi(\mathcal{O})) \geq 2n + 2\chi(\mathcal{O})$ which is $\not\!\!\!$ for $n \gg 0$.

Def $\langle\ ,\ \rangle : \Omega_{K(X)/k} \times R \to k$, $\langle \omega, r\rangle := \sum_{P\in X} \operatorname{Res}_P (r_P\,\omega)$ [makes sense since $r_P\omega \in \Omega_{\mathcal{O}_P/k}$ for almost all P]

Prop 1) $\langle\ ,\ \rangle$ is k-bilinear, 2) $\langle \omega, r\rangle = 0$ if $r \in K(X)$ (by residue formula), 3) $\langle \omega, r\rangle = 0$ if $\omega \in H^0(X,\Omega(-D))$ and $r \in R(D)$ [bec if $v_P(r_P) + v_P(D) \geq 0$ and $v_P(\omega) - v_P(D) \geq 0 \Rightarrow v_P(r_P\omega) \geq 0$], 4) $\langle f\omega, r\rangle = \langle \omega, fr\rangle$ $\forall f \in K(X)$ [clear!].

(γ) **Def** Let $\theta : \Omega_{K(X)/k} \to R^0$, $\theta(\omega) = (r \mapsto \langle\omega, r\rangle)$. By 3) in P $\theta(H^0(X,\Omega(-D)) \subset J(D)$ so in partic $\operatorname{Im}\theta \subset J$. Moreover by 4) in Prop above $\theta : \Omega_{K(X)/k} \to J$ is $K(X)$-linear.

(3) **Prop** $\theta : \Omega_{K(X)/k} \to J$ is injective (hence by (α) it is bijective because both have $\dim_{K(X)} = 1$)

We need

(δ) **L** Let $\omega \in \Omega_{K(X)/k}$ s.t. $\theta(\omega) \in J(D)$. Then $\omega \in H^0(X,\Omega(-D))$.

Pf. Ass not. Then $\exists P \in X$ s.t. $v_P(\omega) - v_P(D) \leq -1$. Put $n = v_P(\omega)$. Define $r \in R$ by $r_Q = 0$ for $Q \neq P$ and $r_P = 1/t_P^{n+1}$. Then $v_P(r_P\omega) = -1$ so $\operatorname{Res}_P(r_P\omega) \neq 0$ so $\langle\omega, r\rangle \neq 0$. Now $r \in R(D)$ because $v_P(r_P) + v_P(D) = -n-1 + v_P(D) = -v_P(\omega) - 1 + v_P(D) \geq 0$. By hyp $\langle \omega, \cdot\rangle$ vanishes on $R(D)$ hence on r $\not\!\!\!$.

Proof of Prop (3) Ass. we have $\omega \in \Omega_{K(X)/k}$ s.t $\theta(\omega) = 0$. In partic $\theta(\omega) \in J(D)$ $\forall D$. By (L) $\Rightarrow$ $\omega \in H^0(X,\Omega(-D))$ for all D $\Rightarrow \omega = 0$.

Prop $\theta : H^0(X,\Omega(-D)) \to J(D)$ is a bij.

Pf. θ well def by (γ). θ inj by (3). It is surj by (L)

Cor $\dim H^1(X,\mathcal{O}(D)) = \dim R/(R(D)+K(X)) = \dim J(D) = \dim H^0(X,\Omega(-D)) = \dim H^0(X,\mathcal{O}(K-D))$. By Lect 15 this proves RR.

Curves $17\frac{1}{2}$ Finite dimensionality of $H^1(X,\mathcal{O})$:

X non-sing proj curve $/k=\bar{k}$, $D \in \mathrm{Div}(X)$.

Recall : $H^1(X,\mathcal{O}(D)) \simeq R/(R(D)+K)$, $K=K(X)$.

(L) If $D \geq E$ then $\deg D - \deg E = \dim \frac{R(D)+K}{R(E)+K} + \ell(D) - \ell(E)$.

Pf. $0 \to \mathcal{O}(E) \to \mathcal{O}(D) \to \text{sky} \to 0 \Rightarrow 0 \to L(E) \to L(D) \to k^{\deg D - \deg E} \to \frac{R}{R(E)+K} \to \frac{R}{R(D)+K} \to 0$

(T) $H^1(X,\mathcal{O})$ is fin dimensional.

(C) $\forall D$, $H^1(X,\mathcal{O}(D))$ is fin dim (cf (T) & (L))

So $y \in L(\Delta)$, $\deg\Delta = d$

Pf of T let $y \in K \setminus k$, $y: X \to \mathbb{P}^1$, $\Delta = y^*(\infty) = (y)_\infty = \sum e_i P_i$. let $A = \mathcal{O}(\mathbb{A}^1)$, $B = \mathcal{O}(X \setminus \Delta)$, $z_1, \dots, z_d$ be basis of $K(X)/K(\mathbb{P}^1)$ (♡). $\exists\, n_0 \in \mathbb{Z}$ s.t $z_j \in L(n_0\Delta)$

Let $n >> n_0$. For $\forall\ 0 \leq s \leq n-n_0$, $y^s z_j \in L(n\Delta)$. So $\ell(n\Delta) \geq (n-n_0+1)d$.

Set $N_n = \dim \frac{R(n\Delta)+K}{R(0)+K}$, By (L) (w/ $D = n\Delta$, $E=0$) get

$$nd = N_n + \ell(n\Delta) - \ell(0) \geq N_n + (n-n_0+1)d - 1$$

so $N_n \leq n_0 d - d + 1$. For $\forall$ div D set $r(D) = \deg D - \ell(D)$. then (for $D \geq E$) by (L) ✿ $r(D) - r(E) = \dim \frac{R(D)+K}{R(E)+K} \geq 0$. Setting $D = n\Delta, E=0$ get $r(n\Delta)$ bounded for $n \to \infty$. Let B be any divisor & take $z \in k[y]$ having high ord zeroes @ all pts of $|B| \setminus |\Delta|$. Then $\exists\, n$, $(z) + n\Delta \geq B$ so

$$r(B) \leq r((z)+n\Delta) \underset{\substack{\uparrow \\ r \text{ dep only} \\ \text{on class in Cl}}}{=} r(n\Delta) \leq C (= \text{const indep of } n)$$

So $\operatorname{Sup}\{r(B) \mid B \in \mathrm{Div}(X)\} < \infty$.

(Lang says: the whole thing is of course pure magic!

By ✿ (w/ $D=B$, $E=0$) $\Rightarrow \dim \frac{R(D)+K}{R(0)+K}$ is bounded as D varies, Since $\bigcup_D R(D) =$

$\Rightarrow \dim \frac{R}{R(0)+K}$ finite $\Rightarrow H^1(X,\mathcal{O})$ fin dim.

(♡) Multiplying z_j by an elt of A may ass $z_j \in B$