Algebraic Curves, II

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(18) Weil's Theorem on normal laws & the Jacobian

(D) Let V be a variety $/ k = \bar{k}$. A normal law on V is a rat map $\mu: V^2 \dashrightarrow V$, $\mu(x,y) =: xy$ s.t $x(yz) = (xy)z$ whenever both defined & the maps $V^2 \dashrightarrow V^2$ $(x,y) \overset{\phi}{\mapsto} (x,xy)$ & $(x,y) \overset{\psi}{\mapsto} (y,xy)$ are birat. (Then using ϕ^{-1}, ψ^{-1} one defines rat maps $V^2 \dashrightarrow V$ den by $(x,y) \mapsto x^{-1}y$ & $(x,y) \mapsto yx^{-1}$) (Say xy defined if $(x,y) \in \operatorname{dom}(\mu)$; simil say $x^{-1}y$ defined, $z(x^{-1}y)$ defined etc.) Gr chunk is a pair (V,μ)

(D) Isomorphism (of gr chunks) $(V,\mu) \to (V',\mu')$ is a bir map $V \dashrightarrow V'$ s.t....

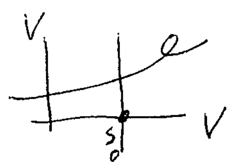
(R) Any alg gr is a gr chunk.

(T) Any gr chunk is iso to a unique alg. gr.

To pv need prep. Let (V,μ) be a gr chunk

(D) An open set $U \subset V$ is good if ($\neq \emptyset$ &) $\forall s \in U$ $\exists a \in U$, s.t $sa, s^{-1}a$ (as, as^{-1}) are defined

(EG) V itself may not be good b/c $\underbrace{V^2 \setminus \operatorname{dom}(\mu)}$ may contain, say verticals $s_0 \times V$ hence $\forall a \in V$, $s_0 a$ not defined.



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(L) $\exists$ good open set $U \subset V$ [Pf: (EX). Idea: rm s_0's as in ↑]

(L) U good $\Rightarrow \forall x,y \in U$ $\exists a$, $x^{-1}(ya)$ etc are defined [Pf (EX). Idea: if $\underbrace{a \text{ suff gen} \Rightarrow ya \text{ suff gen}}$]

Proof of (T): For $\forall s \in U$ Let $U_s = U$. For $\forall s,t \in U$ set $W_{s,t} = \{a \in U \mid s^{-1}a, t^{-1}a \text{ defined}\}$

Let $U_{s,t} = \{s^{-1}a \mid a \in W_{s,t}\}$, $U_{t,s} = \{t^{-1}a \mid a \in W_{s,t}\}$

$\cap$ open U_s $\cap$ open U_t

for uniqueness see (L) on top of p4.

Have
$$\begin{tikzcd} & W_{s,t} \arrow[dl, "s^{-1}\cdot"', "\sim"] \arrow[dr, "t^{-1}\cdot", "\sim"'] & \\ U_s \supset U_{s,t} \arrow[rr, "\sim", "\varphi_{s,t}"'] & & U_{t,s} \subset U_t \end{tikzcd}$$

Define $G = (\coprod U_s)/\sim$ where $\sim$ def by $\varphi_{s,t}$. Let $[x]$ = class of x.

For $a, b \in U$ define $a * b := [b \in U_a]$

(Note: if U = an alg. group then $b \in U_a$ is equivalent to $ab \in U_e$ b/c
so $a*b = [ab \in U_e]$)

$$\begin{tikzcd} & ab \in W_{a,e} \arrow[dl] \arrow[d, "a^{-1}"] \arrow[dr, equal, "id"] & \\ & b \in U_a & U_e \end{tikzcd}$$

Define, for $\alpha = [a_s \in U_s]$, $\beta = [b_t \in U_t] \in G$, the prod $\alpha\beta \in G$ as foll.

Take $x \in U$ suff general. Then $\underbrace{s(a_s x)}_{= a_x}$, $\underbrace{x^{-1}(t\, b_t)}_{= b^x} \in U$ are defined

& we let $\alpha\beta = a_x * b^x = [x^{-1}(tb_t) \in U_{s(a_s x)}]$

(Note: if U = an alg gr we have $\alpha\beta = [\underbrace{a_x\, b^x}_{\in U_e}] = [s\; a_s x x^{-1} t\; b_t] = [s\, a_s t\; b_t \in U_e] = \alpha\beta$

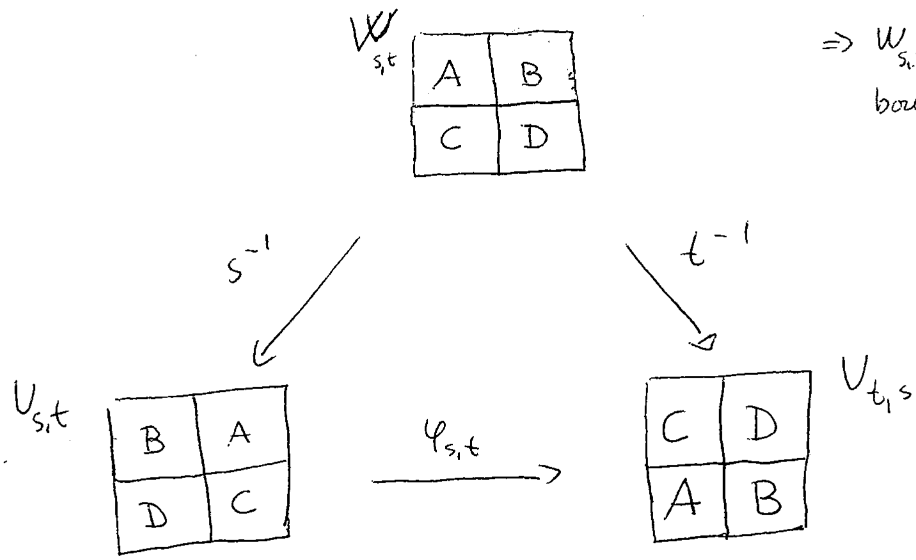
b/c $\alpha = [s a_s \in U_e]$, $\beta = [t b_t \in U_e]$, so $*$ is mult on U)

One checks $(V, \mu) \simeq (G, *)$ & G alg group. (E.g. id elt is $[x^{-1}y \in U_{y^{-1}x}]$ where x, y suff gen)

<u>Picture</u> Say s^{-1} is translation by $\xleftarrow{1/2}$ & t^{-1} is transl by $\downarrow^{1/2}$ on U

& $U \subset \mathbb{T} \setminus F$, $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{C}/\mathbb{Z}[i]$, F = huge fin set on the boundary of the unit square.

$\Rightarrow W_{s,t} = \mathbb{T} \setminus F_1$, $F_1 \subset$ union of boundary $\square$ & of $+$.



(R) Review of finite quotients of $<$ affine / proj varieties based on (L) A f.g k-alg, G fin gr
$G \curvearrowright A \Rightarrow A^G$ f.g.

(R) The symmetric product $X^{(n)}$ of an aff / proj var X

(R) X sm curve $\Rightarrow X^{(n)}$ smooth. (of dim n). Note $X^{(n)} \overset{bij}{\simeq} \{D \in Div(X) \mid D \geq 0,\ deg\, D = n\}$

(L1) X sm proj curve of genus $g \geq 1$. Then, for $\begin{cases} D_0 \in \mathrm{Div}(X), \deg D_0 = 0 \\ D \in X^{(g)}, \text{"suff general"} \end{cases}$
(in a sense clear from pf) we have $\ell(D_0 + D) = 1$ (so $\exists!\ E \geq 0, E \sim D_0 + D$)

Pf. RR $\Rightarrow \ell(D_0 + D) - \underbrace{\ell(K - D_0 - D)}_{i} = g + 1 - g = 1$. So need $i = 0$.

Now $\ell(K - D_0) \underset{RR}{=} 2g-2+1-g+\ell(D_0) = \begin{cases} g-1 \\ g \end{cases}$. Each time I subtract any (except fin many) pt(s) the length of $K - D_0$ drops by 1 (need to avd basepts) (till it reaches 0). So $i = 0$.

(L2) X as above $\Rightarrow X^{(g)}$ is a gr chunk wrt law $D_1, D_2 \in X^{(g)} \mapsto$ unique $E \in |D_1 + D_2 - gP_0|$ (cf (L1)) [where P_0 fixed pt on X]. Write $E = D_1 \circ D_2$

Pf. Need that "suff gen" means $\in$ Zar open.
Need rationality of $(D_1, D_2) \mapsto D_1 \circ D_2$
Need birationality of approp maps; @ least $(D_1, D_2) \mapsto (D_1, E)$ inj b/c if (D_1, D_2') & (D_1, D_2'') have same image $\Rightarrow D_2' \sim D_2'' \underset{(L2)}{\Rightarrow} D_2' = D_2''$

(C) $\exists$ alg gr $J(X)$ iso as a gr chunk with $X^{(n)}$ (so $J(X) \underset{bir}{\sim} X^{(n)}$). called the Jacobian of X. Let $\varphi : X^{(g)} \dashrightarrow J(X)$ be the birat map s.t. $\varphi(D_1 \circ D_2) = \varphi(D_1) + \varphi(D_2)$ for D_i suff gen.

(L1') $\exists$ nat homo $\theta: \mathrm{Div}(X) \longrightarrow J(X)$ s.t. $\theta(D) = \varphi(D)$ for $D \in X^{(g)}$ general. Moreover $\mathrm{Ker}\,\theta = P(X) + \mathbb{Z}P_0$

Pf. Define $\theta(D) := \varphi(N) - \varphi(M)$ where N suff gen & $N \sim D - dP_0 + M$, $d = \deg D$
& check it's a homo. Also by (L1) we have $N = D \circ M$ if $D \in X^{(g)}$ general. So $\theta(D) = \varphi(N) - \varphi(M)$
$= \varphi(D \circ M) - \varphi(M) = \varphi(D) + \varphi(M) - \varphi(M) = \varphi(D)$. Note if $D \sim dP_0$ can tk $N = M$, so $\theta(D) = 0$. Conversely, if $\theta(D) = 0 \Rightarrow \varphi(N) = \varphi(M) \Rightarrow N = M$ b/c φ inj for N gen. $\Rightarrow D - dP_0 \sim 0$

(R). The map $\theta: X \to J(X)$ induces a map $\theta^{(g)}: X^{(g)} \to J(X)$ of sets.

(L3) The map $\theta^{(g)}: X^{(g)} \to J(X)$ coincides with φ.

Pf. Suf $\theta^{(g)}$ & φ coincide on gen pts $M = P_1 + \dots + P_g$. But
$$\theta^{(g)}(M) \underset{\text{def}}{=} \sum_{i=1}^{g} \theta(P_i) \overset{\theta \text{ is homo}}{=} \theta(M) \underset{(L1')}{=} \varphi(M)$$

(L4) $\theta^{(g)}$ surj [Pf: $X^{(g)}$ complete so $\theta^{(g)}(X^{(g)})$ closed in $J(X)$. But $\theta^{(g)}$ bir so dominant & done] (C) $\theta: \mathrm{Div}^0(X) \to J(X)$ surj. (C) $Cl^0(X) \simeq J(X)$ (C) $J(X)$ complete

(L2 1/2) $\theta: X \to J(X)$ is reg (hence $\theta^{(g)}: X^{(g)} \to J(X)$ is reg) [see p-4]

(L5) Let $f: G \dashrightarrow H$ be a rat map b/t 2 alg grs & ass $f(xy)=f(x)f(y)$ for all $x,y \in \underset{\text{open}}{U} \subset \mathrm{dom}(f)$ s.t. $xy\in U$. Then f is reg (& hence ✱ holds everywhere)

Pf. Fix $a\in G$. Claim: $\exists\, x\in U$ s.t. $xa\in U$ (indeed, if $\forall x\in U$ have $xa\subset G\setminus U$ then $Ua\subset G\setminus U$. But Ua & U are open so they meet). By claim $f(x)^{-1}f(xa)$ makes sense & can be taken as def for $f(a)$.

(T) (univ prop of $J(X)$). If $u: X\to A$ mor to A ab var (i.e. compl alg gr) then $\exists!\, v$,

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, "u"'] & J(X) \arrow[d, "v"] \\ & A \end{tikzcd}$$

Pf. u induces $u^{(g)}: X^{(g)}\to A$ so get rat map $J(X) \xrightarrow{\text{bir}} X^{(g)} \xrightarrow{u^{(g)}} A$. One checks v sat hyp of (L) above. So v is a (reg) homo.

(R) $J(X)^{an} = \mathbb{C}^g/\Lambda$ (over $k=\mathbb{C}$) (the latter as in Abel-Jac) [Sketch of pf: Abel-Jac implies $\mathbb{C}^g/\Lambda$ is biner to $X^{(g)}$. So $J(X)^{an}$ & $\mathbb{C}^g/\Lambda$ biner(†). Exactly as in (L5) on top of page they must be iso

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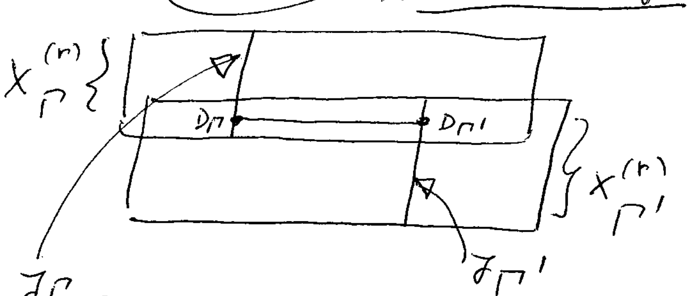
Pf of $(L2\tfrac{1}{2})$ Let $P_1\neq P_0$ (Fix P_1). Then $\exists g$, $(g) = N + P_1 - P_0 - M$, $N\in X^{(g)}$, M suff gen $\in X^{(g)}$.
Fix M. So $(g)_0 \le N+P_1$, $(g)_\infty \le M+P_0$. One can show "=" (Serre p.86). Let $g: X\to \mathbb{P}^1$ so $(g-g(P)) = g^*g(P) - P_0 - M = \underbrace{P+H_P}_{g^*g(P)} - P_0 - M$ so $H_P \sim -P+P_0+M$

so $\underset{-\theta(P)}{\theta(-P)} = \varphi(H_P) - \varphi(M)$, But $P\mapsto \varphi(H_P)$ is regular (Serre)

Another constr of $J(X)$: $r\ge 2g-1$, $\Gamma\in X^{(r-g)}$, RR $\Rightarrow X^{(r)}_\Gamma = \{D\in X^{(r)} \mid \ell(D-\Gamma)=1\}$ open in $X^{(r)}$ & $\bigcup_\Gamma X^{(r)}_\Gamma = X^{(r)}$. Set $D_\Gamma \in |D-\Gamma|$ for $D\in X^{(r)}_\Gamma$; $J_\Gamma = \{D_\Gamma \mid D\in X^{(r)}_\Gamma\}$ closed in $X^{(r)}_\Gamma$
$J_{\Gamma,\Gamma'} := J_\Gamma\cap X^{(r)}_{\Gamma'}$, $\varphi_{\Gamma\Gamma'}: J_{\Gamma,\Gamma'}\to J_{\Gamma',\Gamma}$
$\varphi_{\Gamma,\Gamma'}(D_\Gamma) = D_{\Gamma'}$, $J(X) = \left(\coprod J_\Gamma\right)/\sim$
$\sim$ def by φ



† via a bimero respecting gr laws generically

(19) Topology of complex curves

(D) A top manifold of dim n is a top sp loc homeo to $\mathbb{R}^n$; top surface if $n=2$.

(D) X non-sing var$/_{\mathbb{C}}$ has a can str of top mnfld of dim $2d$ where $d=\dim(X)$. [if $d=1$, $P\in X$ take fund sys of nbhd's t_P^{-1}(disks) where $t_P: U_P \to \mathbb{A}^1$ par.] (EX) $\mathbb{P}^1 \simeq S^2$ (homeo) (+)

(P) X non-sing curve, $P\in X \Rightarrow$ image of $\mathcal{O}_P$ in $\hat{\mathcal{O}}_P = \mathbb{C}[[t_P]]$ is cont in convergent series $\mathbb{C}\{t_P\}$

(P) Any mor of non-sing curves is locally homeo to $z \mapsto z^e$ [If $f: X\to Y$, $f(Q)=P$, $t_P = u\cdot t_Q^e = (\underbrace{u^{1/e}}_{\text{hol branch}} t_Q)^e$]

(D) A triangulation of a Haus. top sp X is a (finite) family Φ of closed subsets E_i of X, homeo's $t_i: E_i \to \sigma_i$ = simplex of dim d_i s.t. 1) $X=\cup E_i$, 2) $E_i \cap E_j = \{\phi$ or E_k, 3) $E_k \subset E_i \Rightarrow$
$\Rightarrow t_i(E_k)$ is a face of σ_i & all faces arise like this.

(D) A combinatorial surface is a top surf X + a triangul Φ s.t. $d_i \le 2$, $d_i=1 \Rightarrow E_i$ is a face of some E_k & $\forall$ 1-simplex is a face of exactly 2 simplices.

(D) A comb. surface is called orientable if $\exists$, for each simplex a cyclic (order of the vertices) of its triangles s.t. $\forall$ edge the induced orient from the 2 adjacent triangles are oposite

(P) (X,Φ) orientable iff $H_2(X,\mathbb{Z})\neq 0$. [Pf skipped] (R) Orientability is indep of Φ.

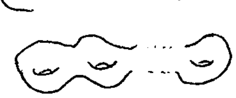
(D) Let (X,Φ) be a combin surface where Φ has c_0 vertices, c_1 edges, c_2 triangles

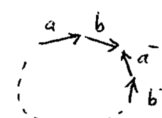
Define $e_\Phi(X) = c_0 - c_1 + c_2$. (Euler characteristic)

(P) $e_\Phi(X) = e(X) := \dim_{\mathbb{R}} H_0(X,\mathbb{R}) - \dim_{\mathbb{R}} H_1(X,\mathbb{R}) + \dim_{\mathbb{R}} H_2(X,\mathbb{R}) =: b_0 - b_1 + b_2$ [Pf: skipped]

(R) $e_\Phi(X)$ does not depend on Φ.

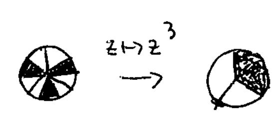
(T) Two (orientable) combinatorial surfaces are homeo iff they have the same Euler char. [Pf skipped]

(P)  is orientable & has Euler char (2-2g where) g = # holes [EX]

(P) $P_{4g}/\sim$ is orientable w/ Euler ch $2-2g$, where P_{4g} =  w/ $4g$ sides. [EX]

(P) For any mor $f: X\to\mathbb{P}^1$ from a non-sing proj curve X $\exists$ compatible (++) comb. surf. structures (Φ & Ψ) on X & $\mathbb{P}^1$. Also both orientable.

Pf. (idea) Reduce to $D\to D$, $z\mapsto z^e$ where it's clear.



when passing from right to left in the disk picture

(R) # of triangles gets multiplied by e | The quantity (# vertices -1) gets multiplied by e.
# of edges —"— —"— | So # vertices on left $= e\times$(# vert on right) $-(e-1)$

(++) i.e. $\forall E\in\Psi$ have $f^{-1}(E) = \cup E_i$, $E_i \in \Phi$, $f/_{E_i}: E_i \to E$ homeo & all simpl in Φ occur like this.

(+) May ass, by projection, $\underbrace{X \subset \mathbb{A}^2}_{P=(0,0)\in}$ given by $f(x,y)=0$ & $t_P = x$ ($f'_x(0,0)\neq 0$). Then $y=y(x)$ holo by implicit fcn thm. So any ratl fcn in x,y is holo. in x.

(T) X non-sing proj curve of genus g $\Rightarrow$ $e(X) = 2-2g$ (hence homeo to $P_{4g}/\sim$)

Pf. Choose mor $f: X \to \mathbb{P}^1$ & compat triangul's Φ & Ψ (+) w/ #'s c_0, c_1, c_2 & c_0', c_1', c_2'

So $e(X) = c_0 - c_1 + c_2$, $2 = e(\mathbb{P}^1) = c_0' - c_1' + c_2'$. Write $f^*P = \sum_{Q \mapsto P} e_Q Q$ (++)

Taking $\sum_{Q \mapsto P}$ in last (R) get $c_2 = nc_2'$, $c_1 = nc_1'$, $c_0 = nc_0' - \sum_Q (e_Q - 1)$ (+++)

so $e(X) = 2n - \sum_Q (e_Q - 1) \underset{\text{Hurwitz}}{=} 2-2g$ $\square$

(R) Since $b_0 = b_2 = 1$ (EX) $\Rightarrow$ $b_1 = 2g$

(R) $\forall$ non-sing proj var $X/\mathbb{C}$, if $b_1 = \dim_{\mathbb{R}} H_1(X, \mathbb{R})$, $h^{10} = \dim_{\mathbb{C}} H^0(X, \Omega^1)$ $\Rightarrow$ $b_1 = 2h^{10}$.

<u>Riemann existence Th</u>

(D) A Riem surface is a top surface + open covering + charts to open sets of $\mathbb{C}$ s.t coord changes are holo
A mor of RS is --- (+++)

(P) A Riem surf is orientable (EX) hence $\underbrace{\text{homeo}}_{\text{if compact}}$ to $P_{4g}/\sim$.

(P) X non-sing curve $\Rightarrow$ X has nat str of Riem. surface.

(T) $\forall$ compact RS arises from an alg curve. Also GAGA for { morphisms, rat fcns (mor to $\mathbb{P}^1$), 1-forms.

Pf1 Step 1 $\exists$ non-ct mero fcn (hard analysis: PDE's)
Step 2 the rest (comparatively easier : one needs to pv RR in this context) WILL BE SKIPPED

Pf2. Step 1. Uniformization (hard analysis) (~~will be skipped~~)
Step 2. Constructing automorphic fcns (comparatively easy)
(This WILL be xplained)

<u>Uniformization</u>

(D) Let X be a top mnfld. The univ cover is a cont map $\tilde{X} \xrightarrow{\pi} X$ w/ $\begin{cases} \tilde{X} \text{ simply conn } \& \\ \pi \text{ top cover (\&)} \end{cases}$

(P) $\tilde{X}$ exists & is unique.

(P) Gr of deck transf $\simeq \pi_1(X)$

(P) $\forall$ $Y \to X$ w/ Y simply conn lifts to $Y \to \tilde{X}$.

(R) X complex mnfld $\Rightarrow$ so is $\tilde{X}$. & $X = \tilde{X}/G$, $\pi_1(X) \simeq G < \mathrm{Aut}(\tilde{X})$, G acting freely.

(T) Any connected simply conn RS X is iso to $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{H}$ $\longrightarrow$ (for $X \subset \mathbb{C}$ Riemann, for X arb Köbe)

$\mathbb{H} \simeq \mathbb{D} = \{z \mid |z|<1\}$

$\uparrow$ again, hard analysis

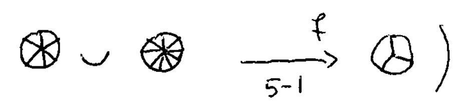
(P) $\begin{cases} \mathrm{Aut}(\mathbb{P}^1) = PGL(2,\mathbb{C}) \\ \mathrm{Aut}(\mathbb{C}) = \{z \mapsto az+b \mid a \in \mathbb{C}^\times, b \in \mathbb{C}\} = \mathbb{C}^\times \ltimes \mathbb{C} \\ \mathrm{Aut}(\mathbb{D}) = \{z \mapsto \theta \frac{z-\alpha}{1-\bar{\alpha}z} \mid |\theta|=1, |\alpha|<1\} \end{cases}$ $\in$ course in $\mathbb{C}$-analysis

(P) Any <u>aut</u> of $\mathbb{P}^1$ has a fixed pt, so no sgr of Aut $\mathbb{P}^1$ $\langle \neq 1 \rangle$ acts freely except $\{1\}$. So if univ cover of X is $\mathbb{P}^1$ then $X = \mathbb{P}^1$

(&) $\forall$ pt $\exists$ nbhd s.t preim is $\coprod$ of copies.

(+++) more generally define complex mnflds.

(+) as in last prop. (++) so $\sum_{Q \mapsto P} e_Q = n := \deg f$ (

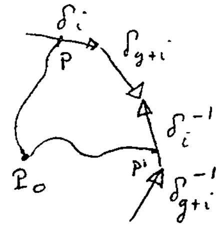


)

$e=2$ $e=3$

(38) Review of Abel's Theorem (+)

Topological review: If X is a cmpct Riem. surface (of genus $g \geq 1$) & $pt \in X$ $\exists$ cycles (loops) $\delta_1, \dots, \delta_{2g}$ based in pt s.t. their complement Δ is simply connected & cutting along loops gives polygon Δ w $4g$ sides as below. Take $P_0 \in X$.



Def: A diff of 1^{st}, 2^{nd}, 3^{rd} kind is a mero 1-form which is holo, has zero res, or has simple poles only. Clearly $1^{st} \Rightarrow 2^{nd}$ & 3^{rd}

(α) Th (Reciprocity law for diff of 1^{st} & 3^{rd} kind) Let ω be of 1^{st} kind & η of 3^{rd}. Then

$$\sum_{i=1}^{g}\left(\int_{\delta_i}\omega \int_{\delta_{g+i}}\eta - \int_{\delta_{g+i}}\omega \int_{\delta_i}\eta\right) = 2\pi\sqrt{-1}\sum_P \mathrm{Res}_P(\eta)\int_{P_0}^{P}\omega$$

Pf. If $P \in \delta_i$, $P' \in \delta_i^{-1}$ corr to eachother, setting $\pi(P) = \int_{P_0}^{P}\omega$, $P \in \Delta$, have $\pi(P') - \pi(P) = \int_{\delta_{g+i}}\omega$. If $P \in \delta_{g+i}$, $P' \in \delta_{g+i}^{-1}$ corr to eachother have $\pi(P') - \pi(P) = -\int_{\delta_i}\omega$. So $\int_{\delta_i + \delta_i^{-1}}\pi\eta = \int_{\delta_{g+i}}\omega\int_{\delta_i}\eta$, $\int_{\delta_{g+i}+\delta_{g+i}^{-1}}\pi\eta = -\int_{\delta_i}\omega\int_{\delta_{g+i}}\eta$

Residue formula $\Rightarrow \int_{\partial\Delta}\pi\eta = \sum \mathrm{Res}_P(\pi\eta) = \sum \mathrm{Res}_P(\eta)\cdot\pi(P)$. Combine w/ $\uparrow$ & done

(Prop) Riemann positivity: if ω is of 1^{st} kind then

$$\sum \sqrt{-1}\left(\int_{\delta_i}\omega \overline{\int_{\delta_{g+i}}\omega} - \int_{\delta_{g+i}}\omega \overline{\int_{\delta_i}\omega}\right) = \sqrt{-1}\int_{\Delta}\omega\wedge\overline{\omega} > 0 \text{ if } \omega \neq 0$$

Pf. Take $\eta = \overline{\omega}$, follow pf of (α) up to resid. formula & instead of res formula apply Stokes $\int_{\partial\Delta}\pi\overline{\omega} = \int_{\Delta} d(\pi\overline{\omega}) = \int_{\Delta}\omega\wedge\overline{\omega}$

(Cor) ω of 1^{st} kind & $\int_{\delta_i}\omega = 0$ $\forall i = 1, \dots, g \Rightarrow \omega = 0$.

So $\exists$ basis $\omega_1, \dots, \omega_g$ (called normalized (of $H^0(X, \Omega^1)$)) s.t. $\int_{\delta_i}\omega_j = \delta_{ij}$ ($1 \leq i,j \leq g$)
(bec $H^0(X, \Omega^1) \to \mathbb{C}^g$, $\omega \mapsto (\int_{\delta_i}\omega)$, is inj hence iso).

(β) (Cor) Let $\omega_1, \dots, \omega_g$ be a normalized basis of $H^0(X, \Omega^1)$. Then

$$\int_{\delta_{g+i}}\omega_j = \int_{\delta_{g+j}}\omega_i, \quad 1 \leq i,j \leq g. \quad \left[\text{Pf: Apply } (\alpha) \text{ to } \omega = \omega_i,\ \eta = \omega_j\right]$$

(+) assumes nothing but definition of Riem. surfaces & the topological description of compact ones. (Also def of $\mathrm{Div}(X)$, $\mathrm{Div}_0(X)$, $P(X) = \{\text{princ. div}\}$)

(γ) Th (Abel) Let $D = \sum n_k P_k$, $\deg D = 0$, be a divisor on X; then D is principal iff $\mu(D) := \left(\sum n_k \int_{P_0}^{P_k} \omega_1, \ldots, \sum n_k \int_{P_0}^{P_k} \omega_g\right) \bmod \Lambda = 0 \in \mathbb{C}^g/\Lambda$, where $\Lambda := \left\{ \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g\right) \mid \gamma \in H_1(X, \mathbb{Z}) \right\} =$

(Cor) μ(+) induces inj map from $Cl(X)$ to $J(X) := \mathbb{C}^g/\Lambda$, easily proved to be surj; Too (surj = Jacobi's th.

Pf. If $D = (f)$ consider $\alpha : \mathbb{P}^1(\mathbb{C}) \to \mathbb{C}^g/\Lambda$, $\alpha([\lambda_0 : \lambda_1]) = \mu((\lambda_0 + \lambda_1 f))$.
where $\mu : \mathrm{Div}_0(X) \to \mathbb{C}^g/\Lambda$, $\mu(\sum m_k P_k) = \left(\sum m_k \int_{P_0}^{P_k} \omega_1, \ldots\right) \bmod \Lambda$. α holo.
But $\forall$ holo $\mathbb{P}^1(\mathbb{C}) \to \mathbb{C}^g/\Lambda$ induces holo bet univ cov $\mathbb{P}^1(\mathbb{C}) \to \mathbb{C}^g$ & any such is ct. So $\alpha([0:1]) = \alpha([1:0])$ & done. For converse need to find η of 3^{rd} kind with poles P_i, $\mathrm{Res}_{P_i} \eta = \frac{1}{2\pi\sqrt{-1}} n_i$ s.t. $\int_{\delta_s} \eta \in \mathbb{Z}$, $1 \le s \le 2g$ [for then $f(P) = \exp\left(2\pi\sqrt{-1}\int_{P_0}^{P} \eta\right)$ well def mero on X & $(f) = \sum n_k P_k$]. Need

(δ) (L) Given fin many pts $P_k \in X$ & $a_k \in \mathbb{C}$ s.t. $\sum a_k = 0$, $\exists\, \eta$ of 3^{rd} kin w/ poles (only) P_k & $\mathrm{Res}_{P_k} \eta = a_k$

Pf. $0 \to \Omega^1 \to \Omega^1(\sum P_k) \xrightarrow{res} \oplus\, \mathbb{C}_{P_k} \to 0 \Rightarrow H^0(\Omega^1(\sum P_k)) \xrightarrow{\rho} \oplus\, \mathbb{C}_{P_k} \to \underbrace{H^1(\Omega^1)}_{\dim 1 \text{ by RR}} \Rightarrow$
$\mathrm{Im}\,\rho$ has cod ≤ 1. But Res theorem says $\mathrm{Im}\,\rho \subset \{(\lambda_i) \mid \sum \lambda_k = 0\}$. So have $=$

Back to Pf of Th (γ)

By (δ) $\exists\, \eta$ of 3^{rd} kind w/ poles P_i & $\mathrm{Res}_{P_i} \eta = \frac{1}{2\pi\sqrt{-1}} n_i$.

Set $\eta' := \eta - \sum_{j=1}^{g} \left(\int_{\delta_j} \eta\right) \omega_j$, get $\int_{\delta_i} \eta' = 0$ for $1 \le i \le g$ & η' has same poles & Res

By Rec law (α) for ω_j, η' have $\int_{\delta_{g+j}} \eta' = 2\pi\sqrt{-1} \sum \frac{n_k}{2\pi\sqrt{-1}} \int_{P_0}^{P_k} \omega_j =$

$$\underset{\substack{\uparrow \\ \text{by hyp} \\ \text{for some } \gamma \in H_1(X,\mathbb{Z}) \\ \text{ind of } j}}{=} \int_\gamma \omega_j \underset{\substack{\uparrow \\ \text{setting} \\ \gamma = \sum_{s=1}^{2g} m_s \delta_s}}{=} \sum_{s=1}^{2g} m_s \int_{\delta_s} \omega_j \underset{\substack{\uparrow \\ \text{normalis.} \\ \text{of } \omega_j}}{=} m_j + \sum_{i=1}^{g} m_{g+i} \int_{\delta_{g+i}} \omega_j \underset{\substack{\uparrow \\ (\beta)}}{=} m_j + \sum_{i=1}^{g} m_{g+i} \int_{\delta_{g+j}} \omega_i$$

Set $\eta'' = \eta' - \sum_{i=1}^{g} m_{g+i}\, \omega_i$.

Get (for $j \le g$) $\int_{\delta_j} \eta'' = \int_{\delta_j} \eta' - \sum_{i=1}^{g} m_{g+i} \int_{\delta_j} \omega_i = -m_{g+j} \in \mathbb{Z}$

$$\int_{\delta_{g+j}} \eta'' = \int_{\delta_{g+j}} \eta' - \sum m_{g+i} \int_{\delta_{g+j}} \omega_i = m_j$$

(+) μ called the Abel map; $\mu : \mathrm{Div}_0(X) \to J(X)$.

(21) Automorphic fcns

(D) Let S be a Riem surface (G gr & $G \curvearrowright S$ adm †), A sys of factors of automorphy (SFA) is an elt in $Z^1(G, \mathcal{O}(S)^\times)$ i.e. a coll of fcns $\varepsilon = \{\varepsilon_g\}_{g\in G}$, $\varepsilon_g \in \mathcal{O}(S)^\times$, $\varepsilon_{g_1 g_2} = g_2^*(\varepsilon_{g_1})\cdot \varepsilon_{g_2}$

(EG1) $G = \langle 1, \tau\rangle$, $S = \mathbb{C}$; $\exists!\, \varepsilon$ s.t $\varepsilon_1 = 1$, $\varepsilon_\tau = e^{-2\pi i z}$; or k-power of this

(EG2) G, S arb (int. case $S = \mathbb{H}$) ; $\varepsilon_g = \frac{dg}{dz}$; or k-th power of this

(D) (automorphic fcns) $A_\varepsilon(S) = \{ f \in \mathcal{O}(S) \mid g^*(f) = \varepsilon_g \cdot f,\ g \in G\}$

(P) $G \curvearrowright S$ & ε as above, $f_0, \dots, f_n \in A_\varepsilon(S)$, Assume

1) $\forall x_1, x_2$ w/ $Gx_1 \cap Gx_2 = \phi$ have $\operatorname{rank}\begin{pmatrix} f_0(x_1) & \cdots & f_n(x_1) \\ f_0(x_2) & \cdots & f_n(x_2)\end{pmatrix} = 2$

2) $\forall x_0$, $\operatorname{rank}\begin{pmatrix} f_0(x_0) & \cdots & f_n(x_0) \\ f_0'(x_0) & \cdots & f_n'(x_0)\end{pmatrix} = 2$

Then $f_0, \dots, f_n$ define an emb $S/G \to \mathbb{P}^n$

[Pf 1) ⇒ inj , 2) ⇒ inj @ level of T_g sp's]

(D) Say $A_\varepsilon(S)$ embeds S/G

(T) Ass G, S, ε as in (EG1), Then 1) $A_\varepsilon(S) = \mathbb{C}\cdot\theta$, $\theta_1(z) = \sum_{m\in\mathbb{Z}} e^{2\pi i (mz + \frac{m(m-1)}{2}\tau)}$

2) $A_{\varepsilon^3}(S)$ embeds S/G

Pf. 1) $f \in A_\varepsilon(S) \Rightarrow f(z+1) = f(z) \Rightarrow f\left(\frac{\log t}{2\pi i}\right) =: \varphi(t)$ well def & hdo on $\mathbb{C}^\times$ $\Rightarrow f(z) = \varphi(e^{2\pi i z}) = \sum_{m\in\mathbb{Z}} c_m t^m$ ($t = e^{2\pi i z}$)

Set $\lambda = e^{2\pi i \tau}$ $\Rightarrow$
$$\left.\begin{aligned} f(z+\tau) &= \sum c_m \lambda^m t^m \\ \| \\ e^{-2\pi i z} f(z) &= \sum c_m t^{m-1} = \sum c_{m+1} t^m \end{aligned}\right\} \Rightarrow c_{m+1} = c_m \lambda^m \Rightarrow c_n = c_0 \lambda^{n+(n-1)+\dots+1} = c_0 \lambda^{\frac{n(n-1)}{2}}$$

(Take prod for $m = 0, \dots, n$)

2) $\forall a, b \in \mathbb{C}$, $\theta_1(z+a)\theta_1(z+b)\theta_1(z-a-b) \in A_{\varepsilon^3}(S)$.

Ass 1) in (P) false. $\Rightarrow \exists z', z''$, $z' - z'' \notin \langle 1, \tau\rangle$, $\exists (\alpha, \beta) \in \mathbb{C}^2 \neq (0,0)$ s.t $\alpha f(z') = \beta f(z'')$ for $\forall f \in A_{\varepsilon^3}$

$\Rightarrow \alpha\, \theta_1(z'+b)\theta_1(z'+a)\theta_1(z'-a-b) = \beta\, \theta_1(z''+b)\theta_1(z''+a)\theta_1(z''-a-b)$, all a & b. Fix a, vary b.

Set $z'+b = z$, $z''-z' = \zeta$ ☉ $\Rightarrow \alpha\, \theta_1(z)\theta_1(z'+a)\theta_1(2z'-a-z) = \beta\, \theta_1(z+\zeta)\theta_1(z''+a)\theta_1(z'+z''-a-z)$

$\Rightarrow \frac{\theta_1(z)}{\theta_1(z+\zeta)} = \text{const}\cdot \frac{\theta_1(z'+z''-a-z)}{\theta_1(2z'-a-z)}$, Choose a s.t the 2 numer / the 2 denom have no common zero.

$\Rightarrow u := \frac{\theta(z)}{\theta(z+\zeta)}$ has no zeroes or poles $\Rightarrow g := \log u$ is defined & hdo (b/c u def on simply connected $\mathbb{C}$)

$\Rightarrow \theta_1(z+\zeta) = e^{g(z)}\theta_1(z) \Rightarrow \begin{cases} g(z+1) = g(z) + 2k\pi i & (*) \\ g(z+\tau) = g(z) + 2\pi i \zeta + 2\ell\pi i & (**)\end{cases} \Rightarrow g'(z)$ has periods 1 & $\tau \Rightarrow g' = \text{const}$ (Liouville)

$\Rightarrow g(z) = cz + d \overset{(*)}{\Rightarrow} c = 2k\pi i$, $\overset{(**)}{\Rightarrow} c\tau = -2\pi i\zeta + 2\ell\pi i$ $\Rightarrow \zeta \in \ell - k\tau$ →←

† free & discrete ☉ so z varies, ζ fixed.

(T) Ass G, S, ε is as in (EG2) with $S = \mathbb{H}$. Then $\exists\, k$ s.t. $A_{\varepsilon^k}(S)$ embeds S/G

Pf. (sketch) One constructs Poincaré series by formula

$$\varphi = \sum_{g \in G} J_g^k\, g^*(h)$$

where $h \in \mathcal{O}(\mathbb{H})$ is bounded. (One proves this unif abs conv on compacts). Also for $\gamma \in G$

$$\gamma^* \varphi = \sum_g \gamma^*(J_g^k)\, \gamma^* g^*(h) = \sum_g J_{g\gamma}^k\, J_\gamma^{-k}\, (g\gamma)^*(h) = \underbrace{J_\gamma^{-k}\, \varphi.}_{\text{So } \varphi \in A_{\varepsilon^{-k}}(S)} \quad \text{Etc (?)}$$

Back to theta

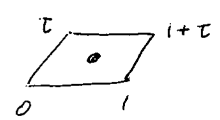
(D) $\theta(z) = \theta(z, \tau) = \theta_1(z + \frac{\tau}{2}) = \sum_{m \in \mathbb{Z}} e^{2\pi i m z + \pi i m^2 \tau}$ (Riemann theta fcn)

$\theta(0, \tau)$ appears in theory to ζ fcn.

(P) (Heat equ) $\dfrac{\partial \theta}{\partial \tau} = \dfrac{1}{4\pi i} \dfrac{\partial^2 \theta}{\partial z^2}$ [Pf obvious]

(P) (modularity) $\theta(z, -1/\tau) = (\tau/i)^{1/2} e^{\pi i z^2 \tau} \theta(\tau z, \tau)$ [Pf: Poisson summation etc ~~For~~ $\cancel{z}=0$ see # theory course]

(P) (zeroes) The zeroes of θ are at $z \equiv \frac{1+\tau}{2} \mod \langle 1, \tau \rangle$

P 

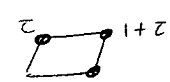
[Pf one computes

$$\Sigma \text{ mult of zeroes of } \theta = \frac{1}{2\pi i} \int_{\partial P} \frac{d\theta}{\theta} = 1$$

$$\Sigma \text{ zeroes of } \theta = \frac{1}{2\pi i} \int_{\partial P} z\, \frac{d\theta}{\theta} \equiv \frac{1+\tau}{2} \mod \langle 1, \tau \rangle]$$

(D) (Weier $\wp$-fcn) $\wp(z, \tau) := \dfrac{1}{z^2} + \sum{}' \left(\dfrac{1}{(z + m\tau + n)^2} - \dfrac{1}{(m\tau + n)^2} \right)$ } mer & $\langle 1, \tau \rangle$-inv

(R) $\wp_z = -2 \sum \dfrac{1}{(z + m\tau + n)^3}$

(P)$^{(*)}$ $\wp_z^2 = 4 \prod_{i=1}^{3} (\wp - e_i(\tau))$, $e_i(\tau) = \wp(\frac{T_i}{2}, \tau)$, $(T_1, T_2, T_3) = (1, \tau, 1+\tau)$ 

(P)$^{(**)}$ $\dfrac{\partial^2}{\partial z^2} \log \theta(z, \tau) = -\wp(z + \frac{1+\tau}{2}, \tau) + \varphi(\tau)$, φ holo.

(*) "EX"; shows again S/G projective, actually a cubic. (**) "EX"

(22) Fuchsian groups

$\Gamma < SL_2(\mathbb{R})$, $\bar{\Gamma} < PSL_2(\mathbb{R})$ its image

$\{\bar{\Gamma} \mid \Gamma \text{ congruence sgr}\} \subset \{\bar{\Gamma} \mid \Gamma \text{ admissible}\} \subset \{\bar{\Gamma} \mid \bar{\Gamma} \text{ arithmetic Fuchsian lattice}\} \subset \{\bar{\Gamma} \mid \bar{\Gamma} \text{ Fuchsian lattice}\}$

congruence sgrs

(D) Γ congr sgr if $\exists N$, $\Gamma(N) < \Gamma < \Gamma(1) = SL_2(\mathbb{Z})$, $\Gamma(N) = \{A \in \Gamma(1) \mid A \equiv I \bmod N\}$

(T) (Wiles et al) $\forall E/\mathbb{Q}$ ell curve $\exists \Gamma$ congr. sgr. & non-ct mor $X_\Gamma \to E$

(Here $X_\Gamma = \mathbb{H}^*/\Gamma$, $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$; NB $\exists$ always cusps!)

admissible sgrs

(D) Γ admissible if $\Gamma < SL_2(\mathbb{Q})$ & $\exists \tau \in M_2^+(\mathbb{Q})$ s.t $\tau \Gamma \tau^{-1} < \Gamma(1)$ w/ fin ind.

(T) (Belyi) $\forall X/\bar{\mathbb{Q}}$ non-sing pr. curve $\exists \Gamma$ admissible s.t. $X \simeq X_\Gamma$

(X_Γ as above ; $\exists$ alw cusps)

Fuchsian lattices

(D) $\bar{\Gamma}$ Fuchs. lattice if discrete & vol (fund domain) $< \infty$ (finite covolume)

(T) (Köbe) $\forall X/\mathbb{C}$ non-sing pr. curve $\exists \bar{\Gamma}$ Fuchs lattice s.t. $Y_\Gamma := \mathbb{H}/\Gamma$ compact (so no cusps!) & $X \simeq Y_\Gamma$.

Arith. Fuchs lattices

(D) Let F be a totally real (recall!) # field, $F \subset \mathbb{R}$ of degree g, $\overset{\text{ind}}{\overset{\shortparallel}{\rho_1}}, \dots, \rho_g : F \to \mathbb{R}$.
Let B be quaternion alg $/F$ (i.e F-alg s.t $B \otimes_F \mathbb{C} \cong M_2(\mathbb{C})$)
Assume $B \otimes_F \mathbb{R} \simeq M_2(\mathbb{R})$, $B \otimes_{\rho_2 F} \mathbb{R} \not\simeq M_2(\mathbb{R})$, etc. (so $B \otimes_{\rho_2 F} \mathbb{R} \simeq$ Hamilton alg etc)
Let $\mathcal{O}_F$ be the ring of int's of F (recall) & $\mathcal{O}_B$ an order of B/F (i.e $\mathcal{O}_F \subset \mathcal{O}_B$ = subring of B
$\mathcal{O}_B$ fin gen $\mathcal{O}$ - module, $\mathcal{O}_B \otimes_{\mathcal{O}} F \cong B$, let $\mathcal{O}_B^1 = \{b \in \mathcal{O}_B \mid \det b = 1\} \subset SL_2(\mathbb{R})$
$\bar{\Gamma}$ called E-arith if it is commensurable (recall) with a conj in $PSL_2(\mathbb{R})$ of a $\overline{\mathcal{O}_B^1}$.

(P) Any E-arith gr is a Fuchs lattice

(D) A Fuchs lattice called arithmetic if E-arith for some E.

(T) ($\Leftarrow$ res of Margulis) Let $\bar{\Gamma}$, $\bar{\Gamma}'$ be Fuchs lattices & $\tau_1, \tau_2 \in PSL_2(\mathbb{R})$ s.t. $\tau_i \bar{\Gamma} \tau_i^{-1} < \bar{\Gamma}'$ w/ fin. index. Ass the corresp $\mathbb{H}/\Gamma' \xleftarrow{\tau_1} \mathbb{H}/\Gamma \xrightarrow{\tau_2} \mathbb{H}/\Gamma'$ has an infinite orbit (xpln). Then $\bar{\Gamma}$ and $\bar{\Gamma}'$ are arith Fuchs lattices.

(23) Belyi's Theorem

(T) Let $X/\mathbb{C}$ be a smooth proj curve. TFAE:

1) X definable $/\bar{\mathbb{Q}}$
2) $\exists\ \Gamma < SL_2(\mathbb{Z})$ of fin index s.t. $X \simeq X_\Gamma$ $(:= \mathbb{H}^*/\Gamma)$
3) $\exists\ f: X \to \mathbb{P}^1$ unr outside $\{0,1,\infty\}$

(R) Pf will show $\forall$ finite $S \subset X(\bar{\mathbb{Q}})$ one can choose $\simeq$ in 2) s.t. $S \subset \{\text{cusps}\}$.
(so Manin-Drinfeld fails for arb Γ).

Proof of T (sketch)

3) $\Rightarrow$ 1) f def $/$ f.g. xtn $K/\bar{\mathbb{Q}} = k$. Embed $k \hookrightarrow \mathbb{C}$. f defines family $F: \mathcal{X} \to \mathbb{P}^1 \times \varphi$ of curves unr outside $\{0,1,\infty\}$. By Riem $\exists$ Th (*) only countably many iso classes in fibers of F so countably many per mat's. Since per mat vary continuously $\Rightarrow$ const. per matrix. Torelli (**) $\Rightarrow$ all fibers iso $\underset{\text{Hilb sch arg}}{\Rightarrow}$ F isotriv $\underset{\text{Hilb}}{\Rightarrow}$ f def $/k$.

2) $\Rightarrow$ 3) $\mathbb{H}^*/\Gamma \to \mathbb{H}^*/SL_2(\mathbb{Z}) \underset{j}{\overset{***}{=}} \mathbb{P}^1$ is unr outside $0, 1728, \infty$. Change coord.

3) $\Rightarrow$ 2) Set $\Gamma_1 = SL_2(\mathbb{Z})$, $\Gamma_2 = [\Gamma_1, \Gamma_1]$ (commutator), One pvs that $\mathbb{H}/\Gamma_2 \cong \mathbb{P}^1 \setminus \{0,1,\infty\}$, $\Gamma_2 = \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})$, $[\Gamma_1 : \Gamma_2] = 12$ (Γ_2 acts freely on $\mathbb{H}$)
($\mathbb{H}$ univ cover of ")
So $\forall$ unr cover of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is $\mathbb{H}/\Gamma$ where $\Gamma < \Gamma_2$ of fin ind (hence $\Gamma < \Gamma_1$ & fin ind.)
So $\forall$ ram — " — $\mathbb{P}^1$ unr outside $\{0,1,\infty\}$ is $\mathbb{H}^*/\Gamma$.

1) $\Rightarrow$ 3) Take $g: X \to \mathbb{P}^1$ def $/\bar{\mathbb{Q}}$ & $S \subset \mathbb{P}^1(\bar{\mathbb{Q}})$ set of crit val's & take $f = \phi \circ g$ where ϕ is as in

(T') Let $S \subset \mathbb{P}^1(\bar{\mathbb{Q}})$ be finite. Then $\exists\ \phi: \mathbb{P}^1 \to \mathbb{P}^1$ def $/\bar{\mathbb{Q}}$, unr outside $0,1,\infty$ (+) s.t. $\phi(S) \subset \{0,1,\infty\}$

Pf. Notation: for $\phi: \mathbb{P}^1 \to \mathbb{P}^1$ let $S_\phi := \phi(S) \cup \overbrace{\{\text{critical values of } \phi\}}^{\phi(\Omega_\phi)}$
$= \phi(S \cup \Omega_\phi)$, $\Omega_\phi = \{\text{crit pts of } \phi\}$

(*) $\forall$ cov of Y unr outside S uniquely det by monodr rep.

(***) one pves this!

(+) i.e. $\phi(\Omega_\phi) \subset \{0,1,\infty\}$.

Case $S \subset \mathbb{P}^1(\mathbb{Q})$.

Ind on $n = |S|$. If $n \le 3$ take ϕ auto. Ass $n \geq 4$ & o.k. for $n-1$. Claim $\exists \phi : \mathbb{P}^1 \to \mathbb{P}^1$ def / $\mathbb{Q}$ s.t. $S_\phi \subset \mathbb{P}^1(\mathbb{Q})$ & $|S_\phi| \le n-1$. Then done (b/c by ind $\exists \phi_1 : \mathbb{P}^1 \to \mathbb{P}^1$ unr outs $\{0,1,\infty\}$ s.t. $\phi_1(S_\phi) \subset \{0,1,\infty\}$. Then if $f = \phi_1 \circ \phi$ get $f(S) = \phi_1(\phi(S)) \subset \phi_1(S_\phi) \subset \{0,1,\infty\}$ & if $A \xrightarrow[\phi]{} B \xrightarrow[\phi_1]{} C \notin \{0,1,\infty\} \Rightarrow B \notin \Omega_{\phi_1}$. Also $B \notin S_\phi$ so $B \notin \phi(\Omega_\phi)$ so $A \notin \Omega_\phi$. So $A \notin \Omega_f$)

Pf of claim. May ass $0, 1, \infty, \alpha \in S$. Seek $\phi(z) = z^A (z-1)^B$, $A, B \in \mathbb{Z}$, $A \neq 0$, $B \neq 0$, $A+B \neq 0$. (This $\Rightarrow \phi(0), \phi(1), \phi(\infty) \in \{0, \infty\}$). If $\gamma \in \Omega_\phi$, $\gamma \neq 0,1,\infty \Rightarrow \phi(\gamma) \neq 0, \infty$ & $\phi'(\gamma) = 0$ so $0 = \frac{\phi'}{\phi}(\gamma)$. But $\frac{\phi'}{\phi} = \frac{A}{z} + \frac{B}{z-1}$. So $\frac{A}{\gamma} + \frac{B}{\gamma - 1} = 0$. So $A\gamma - A + \gamma B = 0$. So $\gamma = \frac{A}{A+B}$. May choose $A, B \in \mathbb{Z}$ s.t. $\alpha = \frac{A}{A+B}$.
Get $\Omega_\phi \subset \{0,1,\infty,\alpha\}$ so $\phi(\Omega_\phi) \subset \{\phi(0), \phi(1), \phi(\infty), \phi(\alpha)\} \subset \{0, \infty, \phi(\alpha)\}$.
Also $\phi(S) = \phi(\{0,1,\infty,\alpha\}) \cup \phi(S') \subset \cup\, \phi(S') \subset \{0, \infty, \phi(\alpha)\} \cup \underbrace{\phi(S')}_{\text{at most } n-4 \text{ elts}}$

$S' = S \setminus \{0,1,\infty,\alpha\}$

So $\#\phi(S \cup \Omega_\phi) \le n-1$.

General case ~~Double~~ ind on (wrt lexi coord) (n,p) where n is the max deg / $\mathbb{Q}$ of elts in S & p is the # of such elts w/ this degree. Say $\alpha \in S$, $\alpha^n + a_1 \alpha^{n-1} + \dots + a_n = 0$ (min equ.) Say n maximal & # of α's w/ this n is p. Take $\phi(z) = z^n + a_1 z^{n-1} + \dots + a_n$. Then ϕ def / $\mathbb{Q}$ & $S_\phi \subset \phi(S) \cup \{\infty\} \cup \phi(S')$ where $S' = \{z \in \bar{\mathbb{Q}} \mid \phi'(z) = 0\}$ (so elts of S' have deg $\le n-1$). Now pts of $\phi(S)$ have deg $\le n$ & # of those of deg exactly n is $\le p-1$ (b/c $\phi(\alpha) = 0$ so α disappears).

If $n = 0$ I'm in case $S \subset \mathbb{P}^1(\mathbb{Q})$ which was proved. So may ass. $n \geq 1$.

(24) Algebraic de Rham Theorem

Recall: For X sm proj curve$/\mathbb{C}$ of genus g, let $\delta_1,\dots,\delta_{2g}$ be the standard basis (of $H_1(X,\mathbb{Z})$)

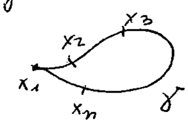
(D) $\Omega^{II}_{\mathbb{C}(X)} = \{$ mero 1-forms of 2nd kind $^{(*)}\} \supset d\Omega_{\mathbb{C}(X)} = \{$exact diff's$\}$

$H^1_{DR}(X) = \Omega^{II}_{\mathbb{C}(X)} / d\Omega_{\mathbb{C}(X)}$

(L.) The map $\Omega^{II}_{\mathbb{C}(X)} \xrightarrow{int} \mathbb{C}^{2g}$, $\omega \mapsto (\int_{\delta_1}\omega, \dots, \int_{\delta_{2g}}\omega)$ has kernel $d\Omega_{\mathbb{C}(X)}$

In partic $\dim H^1_{DR} \le 2g$.

Pf. $\forall$ exact 1-form is in $\operatorname{Ker}(int)$ b/c $\int_\gamma df = f(x_1)-f(x_2)+f(x_2)-f(x_3)+\dots+f(x_n)-f(x_1)$



Conversely if $int(\omega)=0 \Rightarrow \varphi(P) := \int_{P_0}^{P}\omega$ is a well def mero fcn & $\omega = d\varphi$.

(T.) Let $D = P_1 + \dots + P_g$ be non-special. Then the map

$$\left\{\begin{matrix}\text{1-forms of 2nd kind} \\ \text{with polar div} \le 2D\end{matrix}\right\} \xrightarrow{\theta} H^1_{DR}(X) = \frac{\{\text{1-forms of 2nd kind}\}}{\{\text{exact 1-forms}\}}$$

is an iso & dim of both is $2g$. In partic map int in (L) is surjective.

Pf. By (L) Enough to show a) θ inj & b) source of θ has dim $\geq 2g$.

Pf of a). Assume $\theta(\omega)=0$ i.e $\omega = df \Rightarrow f$ has pol. div $\le D$

$\Rightarrow f \in L(D)$. But by RR $\ell(D) = g+1-g+\underbrace{i(D)}_{0} = 1$. So $f \in \mathbb{C}$. So $df=0$.

Pf of b) By RR, $\ell(K+2D) = 2g-2+2g+1-g+\underbrace{\ell(-2D)}_{0} = 3g-1$.

Now source of θ is $\operatorname{Ker}\left(L(K+2D) \to \mathbb{C}^{g-1}\right)$, where map is $\omega \mapsto (\operatorname{Res}_{P_1}\omega, \dots, \operatorname{Res}_{P_{g-1}}\omega)$.

(indeed if $\omega \mapsto 0$ by Res Theorem also have $\operatorname{Res}_{P_g}\omega = 0$ so ω is 2nd kind. The above Ker has dim $\geq \ell(K+2D)-g+1 = 3g-1-g+1 = 2g$ $\square$

(C) $H^1_{DR}(X) \to \operatorname{Hom}(H_1(X,\mathbb{Z}),\mathbb{C}) = H^1(X,\mathbb{C})$, $\omega \mapsto (\gamma \mapsto \int_\gamma \omega)$ is an iso.

Pf Injective by (L). Surj by (T)

(*) i.e w/ zero residues.

(25) Picard-Fuchs Equ

$k=\bar k$ alg closed field, λ variable$/k$, X_λ ell curve $y^2=x(x-1)(x-\lambda)$ over $\overline{k(\lambda)}=k_\lambda$

$\delta=\frac{\partial}{\partial\lambda}: k(\lambda)\circlearrowleft$ xtends uniquely to der $\delta: k_\lambda\circlearrowleft$. Note $K(X_\lambda)=Q\Big(\underbrace{\frac{k_\lambda[x,y]}{y^2-x(x-1)(x-\lambda)}}_{f}\Big)$.

(L) δ xtends uniquely to a k-der $\delta: K(X_\lambda)\circlearrowleft$ s.t. $\delta x=0$

Pf. Def δ on $k_\lambda[x,y]$ s.t. $\delta x=0$, $\delta y=-\frac{x(x-1)}{2y}$. (Then $\delta f=0$ so δ passes to quot.

(L) δ xtends uniquely to a k-der of the $K(X_\lambda)$-module $\Omega_{K(X_\lambda)/k_\lambda}$ s.t. $\delta d=d\delta$.

Pf. Write $\Omega_{K(X_\lambda)/k_\lambda}=\frac{\{\sum a_i\,db_i\}}{\text{relations}}$, xtd δ to numerator (by Leibniz) & show $\delta(\text{den})\subset\text{den}$.

(e.g. $\delta(d(bc)-b\,dc-c\,db) = d(\delta(bc))-\delta b\,dc-b\,d(\delta c)-\delta c\,db-c\,d(\delta b)$

$= d(b\delta c+c\delta b)-\delta b\,dc-b\,d(\delta c)-\delta c\,db-c\,d(\delta b)$

$= \cancel{db\,\delta c}+\cancel{b\,d(\delta c)}+\cancel{dc\,\delta b}+\cancel{c\,d(\delta b)}-\cancel{\delta b\,dc}-\cancel{b\,d(\delta c)}-\cancel{\delta c\,db}-\cancel{c\,d(\delta b)}=0$)

(P) (Fuchs)

$$\omega=\frac{dx}{y}\Rightarrow\quad d\left[\frac{y}{(x-\lambda)^2}\right]=4\lambda(\lambda-1)\delta^2\omega+4(2\lambda-1)\delta\omega+\omega\quad\text{in }\Omega_{K(X_\lambda)/k_\lambda}$$

Pf. (sketch) $\forall$ elt in $K(X_\lambda)$ can be written uniquely as dx times elt in $K(X_\lambda)$.
Do this for LHS & RHS. Start like this

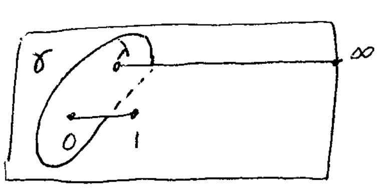
$$y^2=x(x-1)(x-\lambda)\Rightarrow 2y\,dy=[(x-1)(x-\lambda)+x(x-\lambda)+x(x-1)]dx\Rightarrow dy=\frac{\cdots}{2y}dx\ \Downarrow\ \text{LHS}=\ldots$$

$$\overset{\delta}{\Rightarrow}\ 2y\,\delta y=x(x-1)\,\delta(x-\lambda)=-x(x-1)\Rightarrow\delta y=-\frac{x(x-1)}{2y}$$

$$\left.\begin{aligned}\delta\omega&=\delta\Big(\frac{dx}{y}\Big)=dx\cdot\delta\Big(\frac1y\Big)=\cdots\\ \delta^2\omega&=\cdots\end{aligned}\right\}\Rightarrow\ \text{RHS}=\cdots$$

(R) That this miracle should happen xplnd by Gauss-Manin connection.

(R) Say $k=\mathbb{C}$. For $\forall$ value in $\mathbb{C}$ of λ view X_λ as double cover of $\mathbb{C}\cup\{\infty\}$. Let γ on X_λ be as & set $\pi(\lambda)=\int_\gamma\omega$ on X_λ



Then $\frac{\partial}{\partial\lambda}\pi(\lambda)=\int_\gamma\delta\omega$ (b/c γ can be chosen not to mv)

So taking $\int_\gamma$ in (P) get

$$0=4\lambda(\lambda-1)\frac{d^2\pi}{d\lambda^2}+4(2\lambda-1)\frac{d\pi}{d\lambda}+\pi$$

Now $\pi(\lambda)$ bounded as $\lambda\to0$
so $\pi(\lambda)$ is the unique such sol,

$$\pi(\lambda)/\pi(0)=\sum_{n=0}^{\infty}\binom{-1/2}{n}^2\lambda^n$$

(26) Gauss - Manin connection

X non-sing proj curve over alg cl field L

$\Omega^{II}_{K(X)|L} = \{\omega \in \Omega_{K(X)|L} ; \operatorname{res}_P \omega = 0, \text{ all } P \in X\}$ (space of 1-forms of 2nd kind)

Note $\underset{\text{exact forms}}{dK(X)} \subset \Omega^{II}_{K(X)|L}$

(D) $H^1_{DR}(X) = \Omega^{II}_{K(X)|L} / dK(X)$ (deRham space).

(L1) let $\delta : L \to L$ be a der. & $x \in K(X)$ a separ trausc basis for $K(X)|L$

Then $\exists!$ xtn $\delta^x : K(X) \circlearrowleft$ of δ s.t. $\delta^x x = 0$

Also $\exists!$ der $\delta^x : \Omega_{K(X)|L} \circlearrowleft$ commuting w/ $d : K(X) \to \Omega_{K(X)|L}$

Pf (same as in § on Pic-Fuchs)

(L2) y another sep trausc basis. Then for all $f \in K(x)$, $(\delta^y - \delta^x)(f dx) = d(\delta^y_x \cdot f)$.

Pf. $\delta^y - \delta^x$ is an L-der so $\delta^y - \delta^x = u \cdot \frac{d}{dx}$, $u \in K(X) \Rightarrow$

$\delta^y x - \underbrace{\delta^x x}_{0} = u \frac{dx}{dx} = u \Rightarrow (\delta^y - \delta^x)(f dx) = (\delta^y - \delta^x) f \cdot dx + f\, d((\delta^y - \delta^x)x)$

$= u \frac{df}{dx} dx + f du = d(uf)$.

(L3) If $x - x(P)$ is a par @ P then $\operatorname{res}_P(\delta^x \omega) = \delta^x(\operatorname{res}_P \omega)$.

Pf $\omega = (\sum a_n t^n) dt$, $t = x - x(P)$, $a_n \in L \Rightarrow \delta^x \omega = (\sum (\delta^x a_n) t^n + \sum a_n n t^{n-1} \delta^x t) dt$

$+ (\sum a_n t^n) \delta^x dt$.

1st term has residue $\delta^x a_{-1}$

2nd term has residue 0 b/c $\delta^x t = \delta^x(x - x(P)) = -\delta^x(x(P)) \in L$

3rd term is equal to 0 b/c $\delta^x dt = d(\delta^x t) \in dL = 0$.

(L4) If $\omega \in \Omega^{II}_{K(X)|L}$ then $\delta^y \omega \in \Omega^{II}_{K(X)|L}$ for all y.

Pf. Fix P & take x s.t $x - x(P)$ par @ P. Then $\operatorname{res}_P \delta^y \omega = \operatorname{res}_P(\delta^x \omega + \overbrace{(\delta^y - \delta^x)\omega}^{\text{exact by L2}})$

$= \operatorname{res}_P(\delta^x \omega) \underset{L3}{=} \delta^x(\operatorname{res}_P \omega) = \delta^x 0 = 0$

(C) δ^x induces a derivation $\delta : H^1_{DR}(X) \circlearrowleft$ [b/c it preserves 2nd kind & comm w/ d] that does not dep on x [by L2]. Call this Gauss - Manin.

Ⓡ (clio) $H^0(X,\Omega)\to H^1_{DR}(X)$ inj

Pf. If $\omega\in H^0(X,\Omega)$, $\omega = df$, $f\in K(X)$, $f=\sum_{n=\nu}^{\infty} a_n t^n$ @ some P, $a_\nu\neq 0$

$\Rightarrow \nu\geq 0$ (otherwise $\omega = \nu a_\nu t^{\nu-1}+\cdots$ has a pole @ P) $\Rightarrow f\in\mathcal{O}(X)=L \Rightarrow df=0$.

Ⓣ (clio) $\dim_L H^1_{DR}(X)=2g$ (g = genus)

Pf. skip.

Ⓡ Take $\underbrace{\omega_1,\dots,\omega_g}_{\omega}\in H^0(X,\Omega^1)$. Then $\omega, \delta\omega, \delta^2\omega$ linearly dep/L

(⇒ "PicFuchs equ")

(27) The Hasse polynomial & counting pts mod p

Ⓓ $H(\lambda)$ = coeff of x^{p-1} in $[x(x-1)(x-\lambda)]^{\frac{p-1}{2}}$ = coeff of $x^{\frac{p-1}{2}}$ in $[(x-1)(x-\lambda)]^{\frac{p-1}{2}}$

Ⓟ $H(\lambda)=(-1)^{\frac{p-1}{2}}\sum_{r=0}^{\frac{p-1}{2}}\binom{(p-1)/2}{r}^2\lambda^r$

Pf. $H(\lambda)$ = coeff of $x^{\frac{p-1}{2}}$ in $\left\{\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}(-1)^k x^{\frac{p-1}{2}-k}\right\}\left\{\sum_{\ell=0}^{(p-1)/2}\binom{(p-1)/2}{\ell}(-1)^\ell\lambda^\ell x^{\frac{p-1}{2}-\ell}\right\}$

$=(-1)^{\frac{p-1}{2}}\sum_{k+\ell=\frac{p-1}{2}}\binom{(p-1)/2}{k}\binom{(p-1)/2}{\ell}\lambda^\ell$ = desired RHS

Ⓟ $H(\lambda)\equiv \frac{\pi(\lambda)}{\pi(0)}$ mod p in $\mathbb{Z}_p[[\lambda]]$. (where $\pi(\lambda)$ is as before)

Pf. $v_p\left(\binom{-1/2}{n}\right)\geq 1$ for $n\geq\frac{p+1}{2}$ (EX)

For $n\leq\frac{p-1}{2}$, $\binom{-1/2}{n}=\frac{(-\frac12)(-\frac32)\cdots(-(\frac{2n-1}{2}))}{n!}=(-1)^n\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n n!}$

$\equiv\frac{(p-1)(p-3)\cdots(p-(2n-1))}{2^n n!}$

$=\frac{\frac{p-1}{2}(\frac{p-1}{2}-1)\cdots(\frac{p-1}{2}-n+1)}{n!}$

$=\binom{(p-1)/2}{n}$

Ⓓ N_p = #(pts in $\mathbb{F}_p^2$ satisfying $y^2=x(x-1)(x-\lambda)$) + 1

Ⓟ $N_p\equiv 1-H(\lambda)$

Pf. $x(x-1)(x-\lambda)\in\mathbb{F}_p^{\times 2}\iff [x(x-1)(x-\lambda)]^{\frac{p-1}{2}}\equiv1$ or $x=0,1,\lambda$ ($\mathbb{F}_p^\times$ cyclic)

So $N_p-1=\sum_{x\in\mathbb{F}_p}\{1+\underbrace{[x(x-1)(x-\lambda)]^{\frac{p-1}{2}}}_{\sum b_n x^n}\}\underset{\uparrow}{=}-b_{p-1}=-H(\lambda)$

$\sum_{k\in\mathbb{F}_p}x^k\equiv\begin{cases}0 & \text{if } (p-1)\nmid k\\ -1 & \text{if } (p-1)\mid k\end{cases}$

Ⓡ N_p & $\pi(\lambda)$ related. Mystery: what does N_p have to do with $\pi(\lambda)$?

$(*)$ $\frac{dx}{y} = \frac{2dy}{-(3x^2-2(\lambda^p+1)x+\lambda^p)} = \cdots$ on $X^{(p)}$

$= (1+\cdots)dt_p$ @ $(0,0)$

Also $\frac{dx}{y} = \frac{2dy}{-(3x^2-2(\lambda+1)x+\lambda)} = (1+\cdots)dt$ on X, $t = -\frac{2y}{\lambda}$

(28) The Igusa – Manin Theorem

(T) (Igusa–Manin) $X/L = k(\lambda)$ ell curve, $\mathrm{ch}\, k = p > 0$, given by $y^2 = x(x-1)(x-\lambda)$. $\phi : X \to X^{(p)}$ the Frobenius (where $X^{(p)}$ given by $y^2 = x(x-1)(x-\lambda^p)$ & $\phi(\alpha,\beta) = (\alpha,\beta)$ $(``\alpha^p,\beta^p")$). Consider $\phi^* : H^1(X^{(p)},\mathcal{O}) \to H^1(X,\mathcal{O})$, let $\xi \in H^1(X,\mathcal{O})$ be the Serre dual of dx/y & $\xi_p \in H^1(X^{(p)},\mathcal{O})$ the Serre dual of dx/y. Then $\phi^* \xi_p = h \cdot \xi$ ($h \in k(\lambda)$) and $4\lambda(\lambda-1)\delta^2 h + 4(2\lambda-1)\delta h + h = 0$. (I.e. the matrix of Frob on $H^1(\mathcal{O})$ satisfies the P-F equ).

Pf 1 As in Hartshorne compute h explicitly (it's essentially the Hasse polyn) & realize that it satisfies the PF equ. [Mystery persists: this pf is based on "miracle"]

Pf 2 Take $P \in X$, $P = (0,0)$. Also $P^{(p)} = (0,0) \in X^{(p)}$. Then $H^1(X,\mathcal{O}) = \frac{\mathcal{O}(U\cap V)}{\mathrm{Im}(\mathcal{O}(U)\oplus\mathcal{O}(V))}$, $U = X \setminus \infty$, $V = X \setminus P$ & simil $U^{(p)} = X^{(p)} \setminus \infty$, $V^{(p)} = X^{(p)} \setminus P^{(p)}$ [take long ex seq in $0 \to \mathcal{O}_X \to \mathcal{O}_X(\infty P) \oplus \mathcal{O}(\infty 0) \to \mathcal{O}_X(\infty P + \infty 0) \to 0$] Also Serre duality given by $H^1(X,\mathcal{O}) \times H^0(X,\Omega) \to L$, $(\bar a, \omega) \mapsto \mathrm{Res}_P(a\omega)$ for $a \in \mathcal{O}(U\cap V)$. RR $\Rightarrow \exists\, a_p \in \mathcal{O}(U^p \cap V^p)$ with simple pole @ $P^{(p)}$ & res 1 $\Rightarrow a_p = \frac{1}{t_p} + \sum_{n\geq 0} \alpha_n(\lambda) t_p^n$, $t_p = \frac{-2y}{\lambda^p}$. Also write $\omega = \frac{dx}{y} \overset{(*)}{=} (1 + \beta_1^{(p)}(\lambda) t_p + \beta_2^{(p)}(\lambda) t_p^2 + \cdots) dt$

$\omega = \frac{dx}{y} = (1+\beta_1(\lambda)t + \cdots)dt$. Have $\mathrm{Res}_P(a_p\omega) = 1$ so $\xi_p = \bar a_p$. Now ϕ^* sends $t_p \mapsto$ $\bar a_p$ into $a := \frac{1}{t^p} + \sum \alpha_n(\lambda) t^{pn}$.

Also $\mathrm{Res}_P(a\omega) = \beta_{p-1}(\lambda)$, so $\phi^* \xi_p = \beta_{p-1}(\lambda)\cdot \xi$. So enough to show $\beta_{p-1}(\lambda)$ satisfies P-F equ. We know that if $\mathcal{L}^x = 4\lambda(\lambda-1)\delta_x^2 + 4(2\lambda-1)\delta_x + 1$, $\mathcal{L}^t = 4\lambda(\lambda-1)(\delta^t)^2 + \cdots$ then $\mathcal{L}^t\omega = \mathcal{L}^t((1+\beta_1(\lambda)t + \beta_2(\lambda)t^2 + \cdots)dt) =$

$\mathcal{L}^x\omega$ exact by (26) & $\mathcal{L}^x\omega - \mathcal{L}^t\omega$ exact so by (27) $\mathcal{L}^t\omega$ exact so

$= d\left[\sum_{n\geq 0} c_n t^n\right] = \left(\sum n c_n t^{n-1}\right)dt$.

So $L(\beta_n(\lambda)) = (n+1)c_{n+1}$, all n. (where $L = 4\lambda(\lambda-1)\delta^2 + \cdots$)

In partic $L(\beta_{p-1}(\lambda)) = 0$. qed

(R) Pf 2 generalizes to curves of genus $g \geq 1$ (Manin) & has a non-linear generalization (for $g \geq 1$) (B–Voloch)

(R) The fact that matrix of Frob on $H^1(\mathcal{O})$ is related to # of pts in $\mathbb{F}_p$ should be viewed as a Lef fixed pt theorem (de Rham style rather than Betti style)