Analogies b/t numbers & functions

A. Weil's Rosetta stone (from a letter to S. Weil). I will amplify it. [1st & 3rd col appear in Dedekind & Weber + Hensel; middle col might help translation & championed by Artin & Weil]

number fields (number theory)	fction fields in ch p (hybrid)	fction fields in ch 0 (complex geometry)
"curves!" ← i.e. 1-dim theory $\mathbb{Z} \subset \mathbb{Q}$ $\cap \quad \cap$ fin "cts $\Gamma(K)$": $\mu(\mathcal{O}_K) \cup 0 \subset \mathcal{O}_K \subset K$ $\bar{X} \supset X = \operatorname{Spec} \mathcal{O}_K \ni \underline{P}$ $\bar{X} = X \cup \{\underbrace{\tau_1, \dots, \tau_n}\}$, $\tau_i : K \to \mathbb{C}$ max w/ no 2 conjug, $n = r+s$, as usual	curves $\mathbb{F}_p[t] \subset \mathbb{F}_p(t)$ $\cap \quad \cap$ sep fin reg/$\mathbb{F}_p$ $\mathbb{F}_p \subset \mathcal{O}_K \subset K$ $\bar{X} \supset X = \operatorname{Spec} \mathcal{O}_K \ni \underline{P}$ $\bar{X} = X \cup \{\infty_1, \dots, \infty_m\}$ [no picture]	curves $\mathbb{C}[t] \subset \mathbb{C}(t)$ $\cap \quad \cap$ fin $\mathbb{C} \subset \mathcal{O}(X) \subset K$ $\quad \underline{P} \in$ $\bar{X} \supset X$ = aff curve = Spec $\mathcal{O}(X)$ $\bar{X}$ = non-sing. projective curve $= X \cup \{\infty_1, \dots, \infty_n\}$ $\approx$  ∞_i
$\mathbb{Z}_{(p)} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$ = compl of $\mathbb{Q}$ wrt $\lvert\cdot\rvert_p$ $\mathcal{O}_{\underline{P}} \subset \hat{\mathcal{O}}_{\underline{P}} \subset \hat{K}_{\underline{P}} \overset{\text{notn}}{=} K_{\underline{P}} = \cdots$ (*) $\hat{\mathcal{O}}_{\underline{P}} = \{\sum a_i p^i \mid a_i \in \text{cts}(k_{\underline{P}})\}$ ← if $p\mathcal{O}_{\underline{P}} = m(\mathcal{O}_{\underline{P}})$	$\mathbb{F}_p[t]_{\underline{P}} \subset \mathbb{F}_p[[t]] \subset \mathbb{F}_p((t))$ $\mathcal{O}_{\underline{P}} \subset \hat{\mathcal{O}}_{\underline{P}} \subset \hat{K}_{\underline{P}} \overset{\text{notn}}{=} K_{\underline{P}}$ $\hat{\mathcal{O}}_{\underline{P}} = \mathbb{F}_p[[s]]$, $K_{\underline{P}} = \mathbb{F}_p((s))$	$\mathbb{C}[t]_{(t-a)} \subset \mathbb{C}[[t]] \subset \mathbb{C}((t))$ $\mathcal{O}_{\underline{P}} \subset \hat{\mathcal{O}}_{\underline{P}} \subset \hat{K}_{\underline{P}} = K_{\underline{P}} = \mathbb{C}((s))$ $\hat{\mathcal{O}}_{\underline{P}} = \mathbb{C}[[s]]$
For $\underline{P} = \tau_i$, $\hat{K}_{\underline{P}} = K_{\underline{P}} = \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$ ← compl of K wrt $\lvert\cdot\rvert$ (∄ change of chart to mk ∞ look like finite) say $\underline{P} \mid \infty$	For $\underline{P} = \infty_i \mid \infty$ sim pict (change chart.) to mk ∞ look like finite	For $\underline{P} = \infty_i$ similar picture (change chart) to mk ∞ look like finite
Divisors $\underline{P} \nmid \infty \Rightarrow v_{\underline{P}} : K_{\underline{P}} \to \mathbb{Z} \cup \{\infty\}$ (*) $\subset \mathbb{R} \cup \{\infty\}$ $v_{\underline{P}}(\sum_{i \geq m} a_i p^i) = m$, $a_m \neq 0$. $\underline{P} = \tau \mid \infty \Rightarrow v_{\underline{P}} : K_{\underline{P}} \to \mathbb{R} \cup \{\infty\}$ $v_{\underline{P}}(\alpha) = -\log \lvert\tau\alpha\rvert$	Divisors $\underline{P} \nmid \infty \Rightarrow v_{\underline{P}} : K_{\underline{P}} \to \mathbb{Z} \cup \{\infty\}$ $v_{\underline{P}}(\sum_{i \geq m} a_i s^i) = m$, $a_m \neq 0$ $\underline{P} \mid \infty$ same as ↑ in new chart	Divisors $v_{\underline{P}} : K_{\underline{P}} \to \mathbb{Z} \cup \{\infty\}$ same as in col 2

(*) called p unramif. (**) If p unr.
if p ram $v_{\underline{P}}$ chosen to be surj.

top gr (prod top $<$ dis on $\mathbb{Z}^{(\ldots)}$, Eucl on $\mathbb{R}$)

$$\mathrm{Div}(\bar{X}) = \Big\{ \sum_{P \in \bar{X}} \nu_P P \;\Big|\; \nu_P \in \mathbb{Z},\ P \nmid \infty;\ \nu_P \in \mathbb{R},\ P \mid \infty \Big\}$$

(Arakelov div's)

$\mathrm{div}: K^\times \to \mathrm{Div}(\bar{X})$

$\mathrm{div}(f) = \sum v_P(f) P$

$\mathrm{Ker}(\mathrm{div}) = \mu(K)$

$\mathrm{Im}(\mathrm{div}) =: \mathrm{Pr}(\bar{X})$, discr

$Cl(\bar{X}) = \dfrac{\mathrm{Div}(\bar{X})}{\mathrm{Pr}(\bar{X})}$, Haus top gr

(Arakelov classgr)

$\deg: \mathrm{Div}(\bar{X}) \to \mathbb{R}$

$$\deg\Big(\sum n_P P\Big) = +\sum_{P \nmid \infty} n_P f_P \log p + \sum_{P \mid \infty} n_P f_P$$

($f_P = [\frac{\mathcal{O}_K}{P} : \mathbb{F}_p]$, $P \nmid \infty$)

($f_P = 1$, P real)

($f_P = 2$, P complex)

Rem: $D \in \mathrm{Div}(X)$, $a = \prod P^{n_P} \Rightarrow \mathbb{N}(a) := \#(\mathcal{O}_K/a) = e^{\deg(D)}$

(T) (prod formula) $\deg(\mathrm{Pr}(\bar{X})) = 0$

Get $Cl(\bar{X}) \xrightarrow{\deg} \mathbb{R}$

Def $Cl^0(\bar{X}) = \ker(\deg)$

(T) ($\Leftarrow$ Dirichlet Units & Cl th) $Cl^0(\bar{X})$ is compact

(xtn of fin gr $Cl(X)$ by a torus)

$$\mathrm{Div}(\bar{X}) = \Big\{ \sum_{P \in \bar{X}} n_P P \;\Big|\; n_P \in \mathbb{Z} \Big\}$$

$\mathrm{div}: K^\times \to \mathrm{Div}(\bar{X})$

$\mathrm{div}(f) = \sum v_P(f) P$

$\mathrm{Ker}(\mathrm{div}) = \mathbb{F}_p^\times$

$\mathrm{Im}(\mathrm{div}) =: \mathrm{Pr}(\bar{X})$

$Cl(\bar{X}) = \dfrac{\mathrm{Div}(\bar{X})}{\mathrm{Pr}(\bar{X})}$

$\deg: \mathrm{Div}(\bar{X}) \to \mathbb{Z}$

$\deg(\sum n_P P) = \sum n_P f_P$

$\deg(P) = f_P := [\frac{\mathcal{O}_K}{P} : \mathbb{F}_p]$

Rem: for $D = \sum n_P P \in \mathrm{Div}(X)$ & $a = \prod P^{n_P}$ have $\mathbb{N}(a) := \# \frac{\mathcal{O}_K}{a} = p^{\deg(D)}$

(T) (prod for) $\deg(\mathrm{Pr}(\bar{X})) = 0$

Get $Cl(\bar{X}) \xrightarrow{\deg} \mathbb{Z}$

Def $Cl^0(\bar{X}) = \mathrm{Ker}(\deg)$

(T) $Cl^0(\bar{X})$ is finite

$$\mathrm{Div}(\bar{X}) = \Big\{ \sum_{P \in \bar{X}} n_P P \;\Big|\; n_P \in \mathbb{Z} \Big\}$$

$\mathrm{div}: K^\times \to \mathrm{Div}(\bar{X})$

$\mathrm{div}(f) = \sum v_P(f) P$

$\ker(\mathrm{div}) = \mathbb{C}^\times$

$\mathrm{Im}(\mathrm{div}) =: \mathrm{Pr}(\bar{X})$

$Cl(\bar{X}) = \dfrac{\mathrm{Div}(\bar{X})}{\mathrm{Pr}(\bar{X})}$

$\deg: \mathrm{Div}(\bar{X}) \to \mathbb{Z}$

$\deg(\sum n_P P) = \sum n_P$

(T) ($\Leftarrow$ Cauchy res for $\frac{df}{f}$) $\deg(\mathrm{Pr}(\bar{X})) = 0$

Get $Cl(\bar{X}) \xrightarrow{\deg} \mathbb{Z}$

Def $Cl^0(\bar{X}) = \mathrm{Ker}(\deg)$

(T) $Cl^0(\bar{X})$ nat str of ab var (hence as C^∞ mnfld is a torus)

(RR)

Def for $D = \sum \nu_P P \in \mathrm{Div}(\bar{X})$

$H^0(D) = \{ f \in K^* \mid v_P(f) \geq -\nu_P \}$

$$\ell(D) = \log \frac{\# H^0(D)}{2^r (2\pi)^s}$$

$\#\mu(K)$

(RR)

Def for $D = \sum n_P P \in \mathrm{Div}(\bar{X})$

same as in col 3

(Riem-Roch)

Def for $D = \sum n_P P$

$L(D) = H^0(D) = \{ f \in K^* \mid v_P(f) \geq -n_P \}$

$\ell(D) = \dim_{\mathbb{C}} H^0(D)$

$A \to B$ arrow means A inspired B

T (Lang's RR) For $D \in \mathrm{Div}(\bar{X})$

$$\ell(D) = \deg(D) + \ell(0) - g + i(D)$$

where $g = \log \dfrac{\#\mu(K)\sqrt{|d_K|}}{2^r (2\pi)^s}$ &

$$i(D) = O\left(e^{-\frac{1}{n}\deg(D)}\right)$$

(so $i(D) \to \infty$ for $\deg(D) \to \infty$)

(so g deserves name of "genus" given by A. Weil)

T Same as in col 3

T (RR) For $D \in \mathrm{Div}(\bar{X})$

$$\ell(D) = \deg(D) + 1 - g + i(D)$$

where $i(D) = \ell(K-D)$

& $i(D) = 0$ for $\deg D \to \infty$

(K = can divisor, recall)

Def. For $K \overset{\text{Hur}}{\subset} L$ define the

different $\mathcal{D}_{L|K} = \mathrm{Ann}_{\mathcal{O}_L} \Omega_{\mathcal{O}_L / \mathcal{O}_K} \subset \mathcal{O}_L$

& discrim. $\partial_{L|K} = N_{L|K} \mathcal{D}_{L|K} \subset \mathcal{O}_K$

(coincides w/ def in "Alg # Th" for $K=\mathbb{Q}$)

& ramification div $R_{L|K} = \mathrm{div}(\mathcal{D}_{L|K})$

(where $\mathrm{div}(\prod \mathfrak{P}^{n_\mathfrak{P}}) = \sum_{\mathfrak{P} \nmid \infty} n_\mathfrak{P} \mathfrak{P}$)

& canonical div $R = R_L = R_{L|\mathbb{Q}}$

T $\deg R = 2g - 2\ell(0)$

T (Hur) For $K \subset L$ of deg d

$$2g(L) - 2 = d(2g(K) - 2) + \deg R_{L|K}$$

Hurwitz

T sim

T sim

Hurwitz

(*) i.e. $\forall$ compact K $\#\{\gamma \in \Gamma \mid \gamma K \cap K \neq \phi\} < \infty$

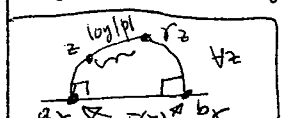
(**) all elts $\gamma \in \Gamma$ have $(\mathrm{Tr}\,\gamma)^2 > 4$, equivalently γ conj to $z \mapsto pz$. Call $p = N(\gamma)$

(***) γ primitive if it generates the isotropy gr of its fixed set $F(\gamma) = \{a_\gamma, b_\gamma\} \subset \mathbb{R} \cup \{\infty\}$

T $\deg K = 2g - 2$

T (Hur) For $\bar{X} \to \bar{Y}$ of deg d

$$2g(\bar{X}) - 2 = d(2g(\bar{Y}) - 2) + \underbrace{\deg R_{X|Y}}_{\text{ram div}}$$



Zeta (Riem/Dedekind)

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \mathbb{N}(\mathfrak{a})^{-s} = \prod_{\mathfrak{P} \nmid \infty} (1 - \mathbb{N}(\mathfrak{P})^{-s})^{-1}$$

(Dedekind zeta) conv for $\mathrm{Re}\, s > 1$

?

(

Zeta (Artin)

$$\zeta_K(s) = \sum_{\mathfrak{P} \in \bar{X}} (1 - \mathbb{N}(\mathfrak{P})^{-s})^{-1}$$

$$= Z(p^{-s})$$

$$Z(t) = \prod (1 - t^{\deg(\mathfrak{P})})^{-1}$$

T $Z(t) = \exp\left[\sum_{\ell \geq 1} \frac{N_\ell}{\ell} t^\ell\right]$

$N_\ell = \# \bar{X}(\mathbb{F}_\ell)$ ...

$\bar{X} = \mathbb{H}/\Gamma$ Zeta (Selberg) (*)

$\Gamma < PSL(2,\mathbb{R})$, $\Gamma \circlearrowright \mathbb{H}$ discont (*) & hyperbolic (**), $\mathbb{P} = \{\mathfrak{P} \mid \mathfrak{P} \subset \Gamma$ conj class, $\mathfrak{P}$ primitive (***) $\}$

$$\zeta_{\bar{X}}(s) = \prod_{\mathfrak{P} \in \mathbb{P}} \prod_{k=0}^{\infty} (1 - N\mathfrak{P}^{-s-k})$$

R $N\mathfrak{P} = \pm e^{\text{length of geodesic } \overline{a_\gamma b_\gamma}}$ ... $\gamma \in \mathfrak{P}$ (geo in $\bar{X}$)

∃ ∞ dim'l analogues of →

(L) $A \in M_n(\mathbb{C})$ w/ eigv $\lambda_1, \dots, \lambda_n$ (or field Ω of ch 0)

$$\exp\left[\sum_{\ell \geq 1} \frac{Tr(A^\ell)}{\ell} t^\ell\right] = \prod_{i=1}^{n} (1-\lambda_i t)^{-1}$$

?

(Hilbert, Connes, Deninger) conjectural pictures

(T) (Weil conj, xpln history (*))
∃ Ω-vect sp's $H^i(\bar{X}) \circlearrowleft A_i$ s.t (***)
1) $N_\ell = \sum (-1)^i Tr(A_i^\ell)$
2) eigv λ_{ij} of A_i have absval $p^{\frac{i}{2}}$
3) fcnal eqn rel $\zeta(s)$ & $\zeta(1-s)$

(T) (Lefschetz) M comp mnfld
$\phi: M \to M$ cont, $N =$ "# Fix ϕ"
Then $N = \sum_{i=0}^{\dim M} (-1)^i Tr(\phi_* : H^i(X, \mathbb{Q}))$

(T) (Riemann Hecke)
$\xi(s) = \xi(1-s)$, $\xi(s) = \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta_{\mathbb{Q}}(s)$

(T) Selberg trace formula

(C) (**) $Z(t) = \prod_{i,j} (1-\lambda_{ij} t)^{(-1)^{i+1}}$ so
a) Z rat'l so ζ mero on $\mathbb{C}$
b) zer & poles of ζ are on $Re\, s = \frac{i}{2}$

(T) (Riem) $\zeta_{\mathbb{Q}}$ mero on $\mathbb{C}$

(C) (RH) $\zeta_{\mathbb{Q}}(s) = 0 \Rightarrow Re\, s = \frac{1}{2}$
sestrip

?

Iwasawa: action of $G(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ on $C\ell(K(\zeta_{p^\infty}))$.

Weil: action of Frob ∈ $G(K\bar{\mathbb{F}}_p^a/K)$ on Jacobian $C\ell^0(K\bar{\mathbb{F}}_p^a)$

Action of $\mathcal{O}_K$ on alg grs (ell vs) or for grs (Lubin-Tate)

Action of $\mathcal{O}_K$ on appropr objects (Carlitz - Drinfeld theory)

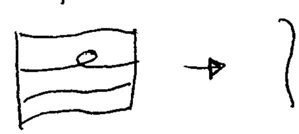
?

2-dim th	2-dim th	2-dim'l theory

2 analogues

analogue w/ $\mathbb{C}$ replaced by $\mathbb{F}_p$
eg... $\mathbb{F}_p(t_1, t_2)$

$f: \bar{\mathcal{S}} \to C = \bar{X}$, $\mathcal{S} = f^{-1}(X)$
2-dim var/$\mathbb{C}$ (surface)   1-dim var/$\mathbb{C}$ (curve)



(I) $\mathcal{S} \to \operatorname{Spec} \mathcal{O}_K = X$
$\hat{\bar{\mathcal{S}}} \to \widehat{\operatorname{Spec} \mathcal{O}_K} = \bar{X}$
$\bar{\mathcal{S}} \setminus \mathcal{S} = \coprod_\tau \mathcal{S}_\tau$
(Arakelov theory)

eg... $\mathbb{C}(t_1, t_2)$

(T) (Mord conj / Falt th) $\# \mathcal{S}(K) < \infty$ if $g \geq 2$

(T) (Samuel)

(T) (Manin'n Mordell over ff,
$\# \mathcal{S}(K) < \infty$ if $g(\text{fib}) \geq 2$
& non-isotriv.

(II) "$\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$" as analogue of $\mathbb{F}_p[t] \otimes_{\mathbb{F}_p} \mathbb{F}_p[t] = \mathbb{F}_p[t_1, t_2]$
(fully conjectural)

(*) Weil in dim 1, Groth-Del arb dim
(**) valid for any dim
(***) Ai induced by Frobenius

Proof: uses deriv's on K = $\mathbb{C}(C)$

arb dim	arb dim	arb dim

(T) (Faltings' Lang)   (T) (Hrush's Lang)   (T) (AB: Lang's conj)

Arakelov int. theory

Metrized line bdle on $\bar{\mathcal{G}}$ means $\bar{\mathcal{L}} = (\mathcal{L}, \rho^{\mathcal{L}})$ with $\mathcal{L}$ lb on $\mathcal{G}$ & $\rho^{\mathcal{L}}$ collection of Herm metrics on $\mathcal{L}_{\mathbb{C},P}$, $P \mid \infty$.

$\mathrm{Pic}(\bar{\mathcal{G}}) = \{\text{pairs}\}/\text{iso}$

$\mathrm{Pic}(\bar{\mathcal{G}}) \times \mathrm{Pic}(\bar{\mathcal{G}}) \to \mathbb{R}$

$$(\bar{\mathcal{L}}, \bar{\mathcal{M}}) = \sum_{P \in X} (\bar{\mathcal{L}}, \bar{\mathcal{M}})_P + \sum_{P \mid \infty} (\bar{\mathcal{L}}, \bar{\mathcal{M}})_P$$

If $\mathcal{L}$ has section $s^{\mathcal{L}}$

$\mathcal{M}$ has section $s^{\mathcal{M}}$

s.t $Z(s^{\mathcal{L}})$ & $Z(s^{\mathcal{M}})$ have no common pt / $\mathbb{Q}$ then

$$(\bar{\mathcal{L}}, \bar{\mathcal{M}})_P = \log \# \frac{\mathcal{O}_P}{(s^{\mathcal{L}}, s^{\mathcal{M}})}, \; P \nmid \infty$$

& if $\mathrm{div}(s_Q) = \sum_{P \in X} n_P P$, $Q \mid \infty$

$$(\bar{\mathcal{L}}, \bar{\mathcal{M}})_Q = -\sum_{P \nmid \infty} n_P \log |s^{\mathcal{M}}(P)|_Q$$

$$-\int_{X_Q} \log |s^{\mathcal{L}}|_Q \, c_1(\rho^{\mathcal{M}})$$

(recall: if $h(P) = \frac{|s_U(P)|}{|s(P)|_\rho}$ then $c_1(\rho) = \partial \bar{\partial} h$)

$s_U : U \to \mathbb{C}$ corr to s in a trivialis of L

similar $\leftarrow$

Inters theory

Define $\mathrm{Pic}(\bar{\mathcal{G}})$

$\mathcal{C}\ell(\bar{\mathcal{G}})$

$\mathrm{Pic}(\bar{\mathcal{G}}) \times \mathrm{Pic}(\bar{\mathcal{G}}) \to \mathbb{Z}$

in terms of assoc divs

For $C \pitchfork D$ & $P \in C \cap D$

$(C, D)_P = 1$. In general

$$(C, D)_P = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathcal{G}}}{(f_C, f_D)}$$

numbers	ff over fin fields	complex fcn fields
$\boxed{\pi_1}$	$\boxed{\pi_1}$ (Grothendieck)	$\boxed{\pi_1}$ (Riemann Poincaré)
nothing in this generality ?	(T) similar (Groth) [Proof uses the transcendental case / $\mathbb{C}$.] ←	(T) $\left\{\begin{matrix}\text{isoclses of unramified coverings}\\ \bar{Y}\to\bar{X}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{subgrs of fin index}\\ \text{in } \pi_1(\bar{X})\end{matrix}\right\}$ ↑ Riem. existence Theorem
?	(R) similar but needs "tame ramification" (Groth) ←	(R) Similar stat for $\bar{X}$ repl by X. This amounts to classifying ramified covers $\bar{Y}\to\bar{X}$. which are controlled by
?	(R) Abhyankhar conj	
BIG UNKOWN $G(K^a/K)=?$ (Langlands program) ←	$G(K^a/K)$ ←	$\varprojlim_{S,N} \dfrac{\pi_1(\bar{X}\setminus S)}{N} = G(K^a/K)$ N nor of finite index
$\boxed{\text{Class field th}}$ (Hilbert, …, Artin) →	$\boxed{\text{Class-field th}}$ (Lang) ←	$\boxed{H_1}$ (Poincaré)
(T) $\left\{\begin{matrix}\text{abelian unramified}^{*}\\ \text{xtensions}\\ \text{of } K\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{sgrs of}\\ Cl^{0}(X)\end{matrix}\right\}$	(T) similar ←	(T) $\left\{\begin{matrix}\text{iso clses of unramified}\\ \text{abelian covers}\\ \bar{Y}\to\bar{X}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{sgrs of fin index}\\ \text{in } H_1(\bar{X},\mathbb{Z})\end{matrix}\right\}$
(*) unramified means unram @ both finite & infinite places. A real place is ramified if $\exists$ a complex place above it		$H_1(\bar{X},\mathbb{Z}) \cong H_1(J,\mathbb{Z})$ $\quad J = Cl^0(\bar{X})$
		$\left\{\begin{matrix}\text{isoclses of}\\ \text{unr coverings}\\ J'\to J\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{sgrs of fin}\\ \text{index in}\\ H_1(J,\mathbb{Z})\end{matrix}\right\}$ $\cong$ duality of above $\left\{\begin{matrix}\text{fin sgrs of}\\ J\end{matrix}\right\}$
(T) Similar ←	(T) similar ←	(R) Ramified version of the above

ord diff calc.

$$\delta_p : \mathbb{Z} \to \mathbb{Z}$$

$$\delta_p a = \frac{a - a^p}{p} \quad (\in \mathbb{Z} \text{ by Fer})$$

$$\delta_p(f+g) = \delta_p f + \delta_p g + C_p(f,g)$$

$$\delta_p(fg) = f^p \delta_p g + g^p \delta_p f + p\,\delta_p f\,\delta_p g$$

(p-der)

AB

ord diff calc

$$\delta_t^{(n)} : \mathbb{F}_p[t] \to \mathbb{F}_p[t], \quad n \geq 0$$

$$\delta_t^{(n)} f :\doteq \text{"}\frac{1}{n!}\delta_t^n\text{"} f$$

$$\mathbb{F}_p = \{ f \in \mathbb{F}_p[t] \mid \delta_t^{(n)} f = 0, n > 0 \}$$

$$\delta_t^{(n)}(f+g) = \delta_t^{(n)} f + \delta_t^{(n)} g$$

$$\delta_t^{(n)}(fg) = \sum_{i=0}^{n} (\delta_t^{(i)} f)(\delta_t^{(n-i)} g)$$

(Hasse-Schmidt derivation)

Hasse

ord diff calc

$$\delta_t : \mathbb{C}[t] \to \mathbb{C}[t]$$

$$\mathbb{C} = \{ f \in \mathbb{C}[t] \mid \delta_t f = 0 \}$$

$$\delta_t(f+g) = \delta_t f + \delta_t g$$

$$\delta_t(fg) = f\delta_t g + g\delta_t f$$

(derivation)

Here I am using that p tot ramifies in $\mathbb{Q}(\zeta_{p^n})$

$\boxed{\text{The totally ramified direction}}$ (as a geometric direction)

Let $\Lambda = \mathbb{Z}_p[[T]]$

Let $K_n = \mathbb{Q}_p(\zeta_{p^{n+1}})$, U_n = units in $\mathbb{Z}_p[\zeta_{p^{n+1}}] = \mathcal{O}_{K_n}$, $U_n^1 = \{u \in U_n \mid u \equiv 1 \bmod \pi_n\}$

$\pi_n = \zeta_{p^{n+1}} - 1$, $\zeta_{p^{n+1}}^p = \zeta_{p^n}$, $N_{n,n-1} : K_n \to K_{n-1}$ the norm. Then (π_n) = max in $\mathcal{O}_{K_n}$

$G(K_n/K_0) = \{a \in (\mathbb{Z}/p^{n+1})^\times \mid a \equiv 1 \bmod p\}$, $G(K_n/K_{n-1}) = \{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \mid a \equiv 1 \bmod p^n\}$

{conjugates of $\zeta_{p^{n+1}}$ in K_n/K_{n-1}} $= \{\zeta_{p^{n+1}}^a \mid a \in \mathcal{G}\} = \{\zeta\zeta_{p^{n+1}} \mid \zeta^p = 1\}$

For $f \in \Lambda$, $N_{n,n-1} f(\pi_n) = \prod_{\zeta^p=1} f(\zeta\zeta_{p^{n+1}} - 1) = \prod_{\zeta^p=1} f(\zeta(\pi_n + 1) - 1)$; so $N_{n,n-1}\zeta_{p^{n+1}} = \zeta_p$

(D) $U = \varprojlim \left(U_0^1 \xleftarrow{N_{10}} U_1^1 \xleftarrow{N_{21}} U_2^1 \xleftarrow{N_{32}} U_3^1 \leftarrow \cdots \right) \ni u = (u_n)$

(D) $\tilde{\Lambda} = \{f \in \Lambda^\times \mid f(0) \equiv 1 \bmod p,\ f((1+T)^p - 1) = \prod_{\zeta^p=1} f(\zeta(1+T) - 1)\} < \Lambda^\times$

(T) (Kummer-Coates-Wiles-Coleman) $\exists!$ group iso $U \xrightarrow{\sim} \tilde{\Lambda}$, $u \mapsto f_u$ s.t. $f_u(\pi_n) = u_n$, $n \geq 0$.

Pf. Define homo $\tilde{\Lambda} \to U$, $f \mapsto (f(\pi_n))$

1) Well defined b/c $1 \underset{(p)}{\equiv} f(0) \underset{(\pi_0)}{\equiv} f(\pi_0)$ so $f(\pi_0) \in U_0^1$ &

$$N_{n,n-1} f(\pi_n) = \prod_{\zeta^p=1} f(\zeta(\pi_n + 1) - 1) = f(\underbrace{(1+\pi_n)}_{\zeta_{p^{n+1}}}{}^p - 1) = f(\zeta_{p^n} - 1) = f(\pi_{n-1}).$$

2) Inj b/c of (L) Any $f \in \mathbb{Z}_p[[T]] \setminus 0$ has only fin many roots in $\{x \in \mathbb{C}_p \mid |x| < 1\}$.

(This is a Corollary of "pwei prep th": any $f \in \mathbb{Z}_p[[T]]$ is $f = p^\nu \cdot$ pol $\cdot$ unit)

3) Surj (This is tricky & skip)

Then let $u \mapsto f_u$ be the inverse of $\tilde{\Lambda} \to U$ □

(D) (Kummer-Coates-Wiles) For $k \geq 1$, $\delta_k : U \to \mathbb{Z}_p$, $\delta_k(u) := \left(D^k \log f_u(T)\right)\Big|_{T=0}$.

(where $D = (1+T)\frac{d}{dT}$)

(R) U is a Λ-module as follows: Let $\Gamma_n = G(K_n/K_0)$, $\Gamma = \varprojlim \Gamma_n$, $\mathbb{Z}_p[[\Gamma]] = \varprojlim \mathbb{Z}_p[\Gamma_n]$

Then (T) (see my notes on meas th) $\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]] = \Lambda$ via $\gamma - 1 \leftarrow T$ where $\gamma = (\gamma_n)$, $\langle \gamma_n \rangle = \Gamma_n$. But U is clearly a $\mathbb{Z}_p[[\Gamma]]$-module. [so mod str dep on γ!]

(T) (see Washington p 312) Let γ be as in ↑ viewed as an elt of $1 + p\mathbb{Z}_p$ (i.e. as a Gal elt does $\zeta_{p^n} \mapsto \zeta_{p^n}^\gamma$). Let $h(T) \in \Lambda$, $u \in U$. Then

$$\delta_k(h(T)u) = h(\gamma^k - 1)\,\delta_k(u)$$

Absolute values

(D) An abs val on a field K is a map $|\ |: K \to \mathbb{R}_+$ s.t. $|x| = 0 \iff x = 0$; $|xy| = |x||y|$; $|x+y| \le |x| + |y|$. (EG) $|\ |_p, |\ |_\infty$ on $\mathbb{Q}$ $(|p|_p = 1/p)$

(R) $|1| = 1$, $|-1| = 1$

(R) $(K, |x-y|)$ metric sp; discrete $\iff |\ |$ triv (i.e. $|x| = 1$ for $x \neq 0$).

(D) $|\ |_1$ & $|\ |_2$ equiv if same topol.

(P) The foll are eq: 1) $|\ |_1$ & $|\ |_2$ equiv; 2) $|x|_1 < 1 \iff |x|_2 < 1$; 3) $\exists s \in \mathbb{R}_+$, $|x|_1 = |x|_2^s$.

Proof. $3) \Rightarrow 1$, $3 \Rightarrow 2$ clear. $1 \Rightarrow 2$ b/c $|x| < 1 \iff x^n \to 0$.

$2 \Rightarrow 3$. May ass $|\ |_1, |\ |_2$ non-triv. Let $|x_0|_1 > 1$. Let $x \in K^\times$. So $|x|_1 = |x_0|_1^\alpha$, $\alpha \in \mathbb{R}$.

Let $\mathbb{Q} \ni \frac{m_i}{n_i} \downarrow \alpha \Rightarrow |x|_1 = |x_0|_1^\alpha < |x_0|_1^{m_i/n_i} \Rightarrow \left| \frac{x^{n_i}}{x_0^{m_i}} \right|_1 < 1 \Rightarrow \left| \frac{x^{n_i}}{x_0^{m_i}} \right|_2 < 1 \Rightarrow |x|_2 < |x_0|^{m_i/n_i}$ $\Downarrow$ $|x|_2 \le |x_0|_2^\alpha$

Sim if $\mathbb{Q} \ni \frac{m_i}{n_i} \uparrow \alpha \Rightarrow |x|_2 \ge |x_0|_2^\alpha$. So $|x|_2 = |x_0|_2^\alpha$. So

$\frac{\log |x|_2}{\log |x|_1} = \frac{\log |x_0|_2}{\log |x_0|_1} =: s$. So $|x|_2 = |x|_1^s$. Again since $|x|_2 > 1 \Rightarrow |x|_1 > 1$ get $s > 0$.

(D) $|\ |$ non-arch if $|n|$ bounded for $n \in \mathbb{N}$.

(P) $|\ |$ non-arch $\iff |x+y| \le \max\{|x|, |y|\}$ (hence also $\iff |n| \le 1$ for $n \in \mathbb{N}$ (or $\mathbb{Z}$))

Proof. $\Leftarrow$ clear. "$\Rightarrow$" Let $|n| \le N$ for $n \in \mathbb{N}$. Let $x, y \in K$, $|x| \ge |y|$. Then $|x|^i |y|^{n-i} \le |x|^n$. So $|x+y|^n \le \sum \left|\binom{n}{i}\right| |x|^i |y|^{n-i} \le N(n+1)|x|^n$. So $|x+y| \le N^{1/n}(1+n)^{1/n}|x|$ & let $n \to \infty$

(T) (Ostrowski) $\forall$ abs val (non-triv) on $\mathbb{Q}$ is equiv to $|\ |_p$ (define!) or $|\ |_\infty$ (define!)

Proof. If $\|\ \|$ non-arch $\exists$ prime p s.t. $\|p\| < 1$ (b/c otherwise by FTA, if $\|p\| = 1$ all $p \Rightarrow \|\ \|$ triv). Then $\underline{a} = \{a \in \mathbb{Z} \mid \|a\| < 1\}$ proper ideal in $\mathbb{Z}$ containing $p\mathbb{Z}$ hence $\underline{a} = p\mathbb{Z}$. So if $p \nmid m$, $n = p^e m \Rightarrow \|n\| = \|p\|^e = |n|_p^\alpha$, $\alpha = -\log\|p\|/\log p$. $\Rightarrow$ qed.

Ass $\|\ \|$ arch. Claim: $\forall m, n > 1$ in $\mathbb{Z}$, $\|m\|^{1/\log m} \overset{(*)}{=} \|n\|^{1/\log n}$. Indeed let $m = a_0 + a_1 n + \dots + a_r n^r$, $0 \le a_i < n$, $n^r \le m$. So $r \le \log m / \log n$. Also $\|a_i\| \le \|1 + \dots + 1\| \le n$

So $\|m\| \le \sum_{i=0}^r \|a_i\| \|n\|^i \le \left(1 + \frac{\log m}{\log n}\right) dM$ with $d = \max(\|0\|, \|1\|, \dots, \|n-1\|)$, $M = \max(1, \|n\|^{\log m/\log n})$. Replace m by m^s & take $\sqrt[s]{\ }$. Get $\|m\| \le \left(1 + s\frac{\log m}{\log n}\right)^{1/s} d^{1/s} M$

Let $s \to \infty$. Get $\|m\| \le \max\{1, \|n\|^{\log m/\log n}\}$. Now since $\|\ \|$ non arch $\exists$ $m = m_0$ s.t. $\|m_0\| > 1$. Get $1 < \|m_0\| \le \max\{1, \|n\|^{\log m_0/\log n}\}$ for all $n > 1$. So $\|n\| > 1$ for all $n > 1$. So $\|m\| \le \|n\|^{\log m/\log n}$ (for all $m, n > 1$). Switching m & n get $(*)$. Now fix n & let's take log in $(*)$

If $s = \frac{\log\|n\|}{\log n}$ then $\frac{\log\|m\|}{\log m} = s$. So $\|m\| = e^{\log m \cdot s} = m^s$. $\square$

(T) K a # field $\Rightarrow$ {abs val}/equiv $\cong$ Max $\mathcal{O}_K$ $\cup$ Hom$(K, \mathbb{C})$/conj/roots

(T) $K = k(t) \Rightarrow$ {abs val's triv on k}/eq $\cong$ Max $k[t]$ $\cup$ Max $k[1/t]$ = Max $k[t]$ $\cup$ {1/t pt}