Artin substitution

Let $K \subset L$ be a Galois ext, A Dedekind w/ $\mathrm{Frac}(A) = K$, B = intcl of A in L
Assume B finite A-alg. (so B Dedekind)

(P) $G(L|K)$ acts transitively on set of primes $\underline{P} \subset B$ dividing a fixed prime $p \subset A$.

$\underline{\mathrm{Pf}}$. Ass $\exists\, \underline{P}' \subset B$, $\underline{P}' | p$, $\underline{P}' \neq \sigma \underline{P}$, $\forall \sigma \in G(L|K)$. Chin rem th $\Rightarrow \exists\, \alpha \in B$ s.t. $\alpha \equiv 0 \bmod \underline{P}'$ & $\alpha \equiv 1 \bmod \sigma \underline{P}$, all σ. Set $a = \prod_\sigma \sigma\alpha = N_{L|K}(\alpha) \in A$. Then $a \notin \underline{P}$, $a \in \underline{P}'$ so $\underline{P} \cap A \neq \underline{P}' \cap A$ $\rightarrow\leftarrow$.

(P) Let $\underline{P} \subset B$ lie over $p \subset A$. Then $\frac{A}{p} \subset \frac{B}{\underline{P}}$ is normal (if in addition separable) $G_{\underline{P}} = G(L|K)_{\underline{P}} \xrightarrow{\varepsilon} G\left(\frac{B}{\underline{P}} \Big/ \frac{A}{p}\right)$ surj. where (**)

$\underline{\mathrm{Pf}}$ Set $\tilde{K} = A/p$, $\tilde{L} = B/\underline{P}$ (*). Let $\tilde{\alpha} \in \tilde{L}$, $\alpha \in B$. Let $F(x) = \prod_{\sigma \in G} (x - \sigma\alpha) \in A[x]$. Then $\tilde{F}(x) \in \tilde{K}[x]$ has $\tilde{\alpha}$ as a root & $\tilde{L}$ cont all conj of $\tilde{\alpha}$. So $\tilde{L}|\tilde{K}$ normal. Now we pv surjectivity of ε. Let $\tilde{L} = \tilde{K}(\tilde{\alpha})$; cf th. of prim elt. Chin rem th $\Rightarrow \exists\, \beta \in B$ s.t. $\beta \equiv \alpha \bmod \underline{P}$ & $\beta \equiv 0 \bmod \sigma\underline{P}$ for all $\sigma \in G \setminus G_{\underline{P}}$. Let $F(x) = \prod_{\sigma \in G} (x - \sigma\beta) \in A[x]$. Then all non-zero roots of $\tilde{F}(x) \in \tilde{K}[x]$ have the form $\widetilde{\sigma\beta}$ w/ $\sigma \in G_{\underline{P}}$. So any conj of $\tilde{\alpha} = \tilde{\beta}$ is one of the $\widetilde{\sigma\beta}$ with $\sigma \in G_{\underline{P}}$. So if $\tilde{\sigma} \in G(\tilde{L}|\tilde{K})$, since $\tilde{\sigma}\tilde{\alpha}$ is a conj of $\tilde{\alpha}$, $\exists\, \sigma \in G_{\underline{P}}$ s.t. $\widetilde{\sigma\beta} = \tilde{\sigma}\tilde{\alpha} = \tilde{\sigma}\tilde{\beta}$. So $\varepsilon(\sigma) = \tilde{\sigma}$. $\square$

(**) $G_{\underline{P}} = \{\sigma \in G(L|K) \mid \sigma\underline{P} = \underline{P}\}$ "decomp. group" = stab of $\underline{P}$ in $G = G(L|K)$.

(*) Denote by $\sim$ red mod p & $\underline{P}$ resp.