DIFF ALG & DIO GEO (6 hours)

Aim: Illustrate interaction between
- diff. algebra (Ritt-Kol)
- dio. geometry (Mord-Lang

Main reference: A. B, Annals of Math 136 (1992), pp 557-567

(also: A.B. DADG, Hermann, 1994)

What will be done here: selfcontained proof of a baby ex. of this theory

Plan
1) Review of alg geo (starting w/ def of varieties
   Curves, Ab var's, "Lang Mordell" (statements)
2) Basic concepts of diff alg geo
3) Stat of "Diff alg Lang" & how it implies
4) Proof of $\#$ Δ
5) More

Mention: theory to be described here triggered a very interesting approach to Lang-Mordell by Hrushovski via model theory ($\in$ logic) J of AMS (1996) - I will not explain this

REVIEW OF ALGEBRAIC GEOMETRY

F alg. closed field

X top space. Call X irred if $X = X_1 \cup X_2$, X_i closed $\Rightarrow \exists i$, $X = X_i$

Def A sheaf of F valued fcns on X is a rule $U \mapsto \mathcal{O}(U) \subset \{f : U \to F\}$ (U open, $\mathcal{O}(U)$ subring)

s.t. 1) $V \subset U$ & $f \in \mathcal{O}(U) \Rightarrow f_{|V} \in \mathcal{O}(V)$

2) $U = \cup U_i$ & $f : U \to F$ & $f_{|U_i} \in \mathcal{O}(U_i) \Rightarrow f \in \mathcal{O}(U)$

Def A (naive) ringed space is a pair $(X, \mathcal{O})$, X = top. sp, $\mathcal{O}$ sheaf as A

Def A mor of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a cont map $f : X \to Y$

s.t. $\varphi \in \mathcal{O}_Y(V) \Rightarrow \varphi \circ f \in \mathcal{O}_X(f^{-1}(U))$

Ex $I \subset F[y_1, \dots, y_n]$ ideal $X = Z(I) = \{\alpha \in F^n \mid f(\alpha) = 0, \text{ all } f \in I\}$

Top on X : closed sets are $X \cap Z(J)$, all J

Assume I prime & set $A(X) = F[y_1, \dots, y_n]/I$

$F(X)$ = quot field of $A(X)$

For $\alpha \in X$ set $M_\alpha = (y_1 - \alpha_1, \dots, y_n - \alpha_n)/I \subset A(X)$

Define $\mathcal{O}$ on X as $\mathcal{O}(U) = \bigcap_{\alpha \in U} A(X)_{M_\alpha}$

$(X, \mathcal{O})$ called aff. var.

$\dim X \overset{\text{def}}{=} \text{tr.deg.} F(X)/F$; Ex $I = 0$, $X \overset{\text{not}}{=} \mathbb{A}^n$, $\dim \mathbb{A}^n = n$.

Note Prod of 2 aff var has str of aff var / Note closed subsets are given by ideals, Irred ones (var's)

Def An alg var is a ringed sp. $\underbrace{(X, \mathcal{O})}_{X \text{ irred}}$ s.t. $\exists X = \cup X_i$ (open cov) w/ $(X_i, \mathcal{O}_{|X_i})$ aff v.

Note Prod of 2 var has str of var.

Ex $I \subset F[x_0, \dots, x_n]$ homog ideal, prime. $\mathbb{P}^n = \dfrac{F^{n+1} \setminus \{0\}}{\sim}$, $v \sim \lambda v$, $\lambda \in F^\times$

$X = Z(I) = \{\alpha \in \mathbb{P}^n \mid f(\alpha) = 0, \text{ all } f \in I\}$ as set

X has nat str. of alg var : $X = \cup X_i$, $X_i = X \cap \{\alpha \in \mathbb{P}^n \mid \alpha_i \neq 0\}$

through bij $\{\alpha \in \mathbb{P}^n \mid \alpha_i \neq 0\} \to F^n$, $(\alpha_0, \dots, \alpha_n) \to \left(\frac{\alpha_0}{\alpha_i}, \dots, \frac{\alpha_0}{\alpha_{i-1}}, \frac{\alpha_0}{\alpha_{i+1}}, \dots, \frac{\alpha_0}{\alpha_n}\right)$

A proj var is an X iso to one above. If I gen by pol's in $K[x_0, \dots, x_n]$ say X def/K & define $X(K)$ so $X(F) = X$

Def X alg var. We define tangent var $T(X)$

Assume X affine $= Z(I)$ & set $T(X) = \{(\alpha, \xi) \in F^{2n} \mid \sum \frac{\partial f}{\partial x_i}(\alpha) \xi_i = 0$, all $f \in I\}$

Call X smooth or non-sing if $\dim T_\alpha(X) = \dim X$, all $\alpha \in X$. $T_\alpha(X)$ = preim of α in $T(X)$.

For X general, $X = \cup X_i$ & call X smooth if all X_i are. Then $T(X_i)$ varieties $\Rightarrow T_X$

Def v. field/U is a section of $T(U) \to U$ Not $\Theta(U) = \{$v. fields on $U\}$

Def $\Omega(X)$ (space of 1-forms): elts are (compl) coll $\omega_U : \Theta(U) \to \mathcal{O}(U)$, $\mathcal{O}(U)$-line.

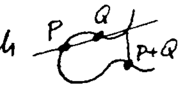
Th X proj(smooth) $\Rightarrow \dim_F \Omega(X) < \infty$.

CURVES, ABELIAN VARIETIES, LANG-MORDELL

Def A curve is a proj smooth var X of dim 1.
Its genus is $g = \dim_F \Omega(X)$; $/\mathbb{C}$ 

Ex: $Z(f) \subset \mathbb{P}^2$, f a homog pol in $F[x_0, x_1, x_2]$ s.t $Z(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}) = \emptyset$.

Def An abel var is a proj var A together w/ a mor $\mu : A \times A \to A$ that sat the group law axioms.

Ex 1) $Z(f)$ as before with $\deg f = 3$ or products of such 

2) $\forall$ curve X of genus g $\exists!$ abel var A of dim g & mor $X \to A$, s.t $\forall$ $X \to B$ ab var $\exists!$ $X \to A$, $\searrow B \downarrow$. A called $\mathrm{Jac}(X)$. (Th) won't need it.

Lang ~~...~~ Conj ((Th) Faltings-Raynaud) (char $F = 0$.)
Let A be an abelian variety, $X \subset A$ curve of genus ≥ 2
$\Gamma \subset A$ subgroup of finite rk (i.e. ~~...~~ $\exists \gamma_1, \dots, \gamma_r \in \Gamma$ s.t $\forall \gamma \in \Gamma$ $\exists$ $m, m_1, \dots, m_r \in \mathbb{Z}$, $m \neq 0$, $m\gamma = \Sigma$...)
Then $X \cap \Gamma$ finite.

Aim here: Prove this in case X is not ~~...~~ iso to a curve def$/\overline{\mathbb{Q}}$ ~~...~~; this is called "the geo case".
Main interest: method $\perp$ usual approaches (Raynaud, Grauert, Manin)
method $\in$ "~~...~~ diff alg geometry".

Standard reduction of geo case (general nonsense)

$$\begin{array}{ccc} \overline{\mathbb{Q}} & \to & F \\ \downarrow & & \downarrow \\ \mathbb{C} & \to & \widetilde{F} \end{array}$$

May assume $\mathbb{C} \subset F$ by enlarging F (f.g. K of tr deg 1), X & A defined by eqns with coeff in F but
X not definable by eqns w/ coeff in $\mathbb{C}$.

Take $\delta : F \to F$ deriv. (i.e $\begin{cases} \text{add map} \\ \text{Leibnitz rule} \end{cases}$ $\begin{cases} \delta(x+y) = \delta x + \delta y \\ \delta(xy) = x\delta y + y \delta x \end{cases}$)

($\delta(K) \subset K$, $\delta(K) \neq 0$) $C := \{x \in F \mid \delta x = 0\} = \mathbb{C}$

Comment (Relashionship w/ Mordell conj)
Mordell Conj X curve of genus ≥ 2 over F, $K \subset F$ subf f.g $/\mathbb{Q}$ (X def $/K$)
Then $X(K)$ finite

All known pfs use $\Leftarrow$ except one here The one here uses $\Rightarrow$	Lang $\xrightarrow{\text{Mordel-Weil}}$ Mordell Falt Raynaud $\Uparrow$ Lang for Γ fin gen $\Leftarrow$ Mord-Weil	Comment on Mazur's question } explain

DIFFERENTIAL ALGEBRAIC GEOMETRY

Assume F alg closed, char $F=0$, $\delta: F \to F$ derivation

$F\{y_1,\dots,y_n\} := F[y_1,\dots,y_n, y_1',\dots,y_n', y_1'',\dots,y_n'',\dots,y_1^{(m)},\dots,y_n^{(m)},\dots]$, $y_i^{(j)}$ new var's

$\exists!\ \delta: F\{y_1,\dots,y_n\} \circlearrowleft$ der extending δ & s.t. $\delta y_i = y_i'$, $\delta y_i' = y_i''$, ...

(Ex of how it works: $\delta(a(y_1')^3) = (\delta a)(y_1')^3 + 3a(y_1')^2 y_1''$)
$(a \in F)$

Note: may evaluate $f \in F\{y\}$ @ $\alpha \in F^n$: $f(\alpha) := f(\alpha, \delta\alpha, \delta^2\alpha, \dots)$.

Fact: May always enlarge F s.t.

$$\forall f, g \in F\{y\},\ f \neq 0,\ \operatorname{ord} f < \operatorname{ord} g\quad \exists \alpha \in F,\ f(\alpha) \neq 0,\ g(\alpha) = 0$$

Call then F "δ-closed" & assume this is the case from now on.
(Not essential but very useful)

Construction of jet spaces X smooth var / F. Will construct an inv-sys of varieties

$$X = J^0(X) \leftarrow J^1(X) \leftarrow J^2(X) \leftarrow \cdots$$

$\cancel{X}\ T(X)$

1) Case X affine, $X = Z(I) \subset \mathbb{A}^n = F^n$, $I = (f_1,\dots,f_m)$

$J^1(X) = Z(I_1) \subset \mathbb{A}^{2n}$, $I_1 = (f_1,\dots,f_m, \delta f_1,\dots,\delta f_m)$

$\delta f = f^\delta + \sum \frac{\partial f}{\partial y_i}$

$J^2(X) = Z(I_2) \subset \mathbb{A}^{3n}$, $I_2 = (f_1,\dots,f_m, \delta f_1,\dots,\delta f_m, \delta^2 f_1, \dots, \delta^2 f_m)$

...

(Note $\exists$ nat derivation δ on $\bigcup \mathcal{O}(J^n(X)) = F\{y_1,\dots,y_n\}/(\delta^i f_j)) \overset{nat}{=} \mathcal{O}(J^\infty(X))$.

2) Case X general – glue. (Have nat δ_U on $\mathcal{O}(J^\infty(U))$, $\forall U \subset X$ affine)

Def "Proalg var / F" is an inv-sys $X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \leftarrow \cdots$ of var's, X_*
s.t. π_i^{-1}(aff) = affine.
A morphism $X_* = (X_i) \to (Y_i) = Y_*$ is an increasing fcn $\tau: \mathbb{N} \to \mathbb{N}$
& system of mor $X_{\tau(i)} \to Y_i$ making appropriate diagr's comm

So X smooth var $\Rightarrow$ $(J^i(X)) = J^*(X)$ proalg var.

Functoriality $X \to Y$ $\Rightarrow$ $J^*(X) \to J^*(Y)$, horizontal (here $\tau = \mathrm{id}$ / later not)

Here, and later, "horizontal" means "compatible w/ δ"

Note If $X = P = Z(y_1 - \alpha_1, \dots, y_n - \alpha_n) \subset \mathbb{A}^n$ is a pt (of Y) $J^m(X) = (y_i - \alpha_i, y_i' - \delta\alpha_i, \dots)$ is a pt. of $J^m(Y)$
$Z(y_i - \alpha_i)$

So get ~~maps~~ maps $\nabla_m: Y \to J^m(Y)$ (NOT morph of varieties)

<u>Def</u> A δ- morphism (of order $\leq m$) is a map (of sets) $X \to Y$ between 2 var's X, Y which factors as

$$X \xrightarrow{\nabla_m} J^m(X) \xrightarrow[\text{var's}]{\text{mor of}} Y$$

A compos of δ-mor is a δ-mor (of ord $\leq$ sum of ord)

$$\begin{tikzcd}
 & J^{m+k}(X) \arrow[r] \arrow[d] & J^k(Y) \arrow[d] \arrow[dl, phantom, "\substack{\text{triangle}\\ \text{not}\\ \text{comm}}"] & & \\
X \arrow[ur, "\nabla_{m+k}"] \arrow[r, "\nabla_m"] \arrow[rr, bend right] & J^m(X) \arrow[r] & Y \arrow[r, "\nabla_k"] \arrow[rr, bend right] & J^k(Y) \arrow[r] & Z
\end{tikzcd}$$

<u>New category</u> objects $= \{$ varieties $/ F \}$ ⎫ its study
morphisms $= \{ \delta$-morphisms $\}$ ⎭ "δ-alg. geo" (*)

<u>Horizontal pro-subv's</u> of $J^*(X)$. By that I mean a seq subv $Z_* = (Z_i)$ s.t. if replace X by aff open sets

$$\begin{array}{ccc} Z_i & \subset & J^i(X) \\ \downarrow & & \downarrow \pi_i \\ Z_{i-1} & \subset & J^{i-1}(X) \end{array}$$

if $I_i =$ ideal of Z_i in $\mathcal{O}(J^i(X))$ then $\delta I_i \subset I_{i-1}$

$I_{i-1} \cap \mathcal{O}(J^{i-1}(X)) = I_{i-1}$ i.e. Z_{i-1} is clos of the image of Z_i

<u>Solution set</u> $Z_*^{sol} = \bigcap_{i=1}^{\infty} \nabla_i^{-1}(Z_i)$

<u>δ- top on X</u> (Kolchin top) Its closed subsets (called δ-closed) are fin unions of sets of the form Z_*^{sol} ~~$\nabla_i^{-1}(Z_i)$~~ not sol, Z_* a hor pro-subv of $J^*(X)$

say who Z_* is.

<u>EX</u> $X = \mathbb{A}^1$, $\{ \alpha \mid (\alpha'')^2 = \alpha^3 \}$ is δ-closed.

Th δ- <u>Nullstellensatz</u> Z_*^{sol} determines Z_*, so $Z_* \to Z_*^{sol}$ is a bij

$$\{ \text{horiz prosubv' of } J^*(X) \} \simeq \{ \text{irred } \delta\text{-closed subsets of } X \}$$

Pf: gen non-sense
Rem: This is not absolutely necessary: only simplifies presentation of arg's.

<u>abs dimension</u> of an irred δ- closed set $\Sigma \subset X$ $\left(\Sigma = Z_*^{sol} \right)$.

<u>EX</u> ...

$$a(\Sigma) = \sup (\dim Z_i) \in \mathbb{N} \cup \{\infty\}$$

$$\left(\text{Note } a(X) = \infty \text{ if } \dim X \neq 0. \right)$$

(*) Actually a larger cat is even more interesting but mysterious: $\{\delta$-varieties$\}$. Define them after ...

DIFFERENTIAL ALGEBRAIC LANG (cf B, 1992 Annals of Math)

F δ-closed, $C = \{x \in F \mid \delta x = 0\}$ const. subfield

"hard one" → **Th1** Let A be an abel var, $X \subset A$ curve of genus ≥ 2 (not iso to one def over C), $\Sigma \subset A$ a δ-closed subgroup, $a(\Sigma) < \infty$. Then $X \cap \Sigma$ finite. [Diff Alg Lang] (Actually holds w/ $\forall$ alg gr in place of A)

"easy one" → **Th2** Let A be an abel var, $\Gamma \subset A$ subgr of fin rk, $\Sigma = \delta$-clos of Γ. Then $a(\Sigma) < \infty$.
(Actually holds w/ $\forall$ comm alg gr in place of A).

Cor A ab var, X curve of genus ≥ 2 (not iso to one def/C), $\Gamma \subset A$ subgr of fin rk. Then $X \cap \Gamma$ finite.
(Lang (geometric))

Pf of Thm 2 (sketch; details easy to fill in; refs for alg gps: – Serre.)

- By fctoriality $J^n(A)$ are commut. alg. groups (var & gr s.t mult is a mor, def) so $\mathrm{Ker}(J^{n+1}(A) \to J^n(A))$ are comm. alg groups.
- $\forall$ smooth var Y, fibres of $J^{n+1}(Y) \xrightarrow{\pi} J^n(Y)$ are $\simeq \mathbb{A}^d$. Indeed if $Y = Z(I)$, $I = (f_1, \dots, f_m)$ then $\delta^{n+1} f_1, \dots, \delta^{n+1} f_m$ are linear in $y_1^{(m)}, \dots, y_n^{(m)}$ so fibres of π are linear subsp. of $\mathbb{A}^n$. [Ex $\delta f = f^\delta + \sum \frac{\partial f}{\partial y_i} \cdot \underline{y_i'}$
$\delta^2 f = f^{\delta^2} + 2\sum (\frac{\partial f}{\partial y_i})^\delta y_i' + \sum \frac{\partial^2 f}{\partial y_i \partial y_j} y_i' y_j' + \sum \frac{\partial f}{\partial y_i} \underline{y_i''}$ etc
- $\forall$ comm alg gr $\simeq \mathbb{A}^d$ is iso to the add gr of $\mathbb{A}^d$
- So: $\mathrm{Ker}(J^{n+1}(A) \to J^n(A))$ add. groups $\mathbb{G}_a^d$
- $\forall$ extension of add gr's is add gr (char 0) well known, see Serre (Lazard), called $\mathbb{G}_a^d$
- so $\mathrm{Ker}(J^{n+1}(A) \to A) \simeq \mathbb{G}_a^N$, so $J^{\infty}(A)$ is an ext of A by "$\mathbb{G}_a^\infty$" (all proj lim of $\mathbb{G}_a^{d_n}$'s, of genuine ∞ dim, are iso to each other; call them $\mathbb{G}_a^\infty$)
- Any ext of ab var A by a $\mathbb{G}_a^\infty$ is an ext of $\mathbb{G}_a^\infty$ by G (an alg gr) (an ext of A by some $\mathbb{G}_a^d$) (foll from Serre)

[cf. May assume proj & take Jac's] $\Rightarrow \mathcal{X}_b$ iso to each other

would this work

(6)

- get $0 \to G_n \to J^n(A) \to \mathbb{G}_a^{d_n} \to 0$, $d_n \to \infty$

  $\mathcal{O}(J^n(A)) = \mathcal{O}(\mathbb{G}_a^{d_n})$ — $\dim G_n$ bounded by, say, e

  So $G_* = (G_n)$ horiz. subproalgvariety, G_n stationary
- get δ-morphism $A \xrightarrow{\Psi} \mathbb{G}_a^{d_{n_0}}$ whose kernel (G_*^{sol}) has abs dim $\le e$

(Remark: Ψ "=" "some non-lin diff eqns"; around 0 they "are" lin & they are Pic-Fuchs eqns)

- $\forall$ fin rk subgr $\Delta \subset \mathbb{G}_a^d$ $\exists$ δ-homo, $\Lambda : \mathbb{G}_a^d \circlearrowleft$, $\Lambda(\Delta) = 0$

  Ker Λ of fin abs dim.

  ~~while~~ [easy; ex: $\Delta \subset \mathbb{G}_a = F$, $\Delta = \langle \eta_1, \eta_2 \rangle$, say, $\eta_i \in F$ lin ind,

  Let $\Lambda(y) = y'' + ay' + by$ w a & b s.t $\begin{cases} \eta_1'' + a\eta_1' + b\eta_1 = 0 \\ \eta_2'' + a\eta_2' + b\eta_2 = 0 \end{cases}$
- By previous rem $\exists$ $\Lambda : \mathbb{G}_a^{d_{n_0}} \circlearrowleft$ s.t $\Lambda(\Psi(\Gamma)) = 0$

  So $A \xrightarrow{\Psi} \mathbb{G}_a^{d_{n_0}} \xrightarrow{\Lambda} \mathbb{G}_a^{d_{n_0}}$ is s.t its kernel has fin. abs dim & contains Γ.

$\square$

PREPARATION FOR THM I : SPREADING OUT

$\begin{matrix} \mathcal{X} \\ \downarrow \\ \mathcal{B} \end{matrix}$ mor of var / $\mathbb{C}$, $K = \mathbb{C}(\mathcal{B})$, $F = \overline{K}$ $\Rightarrow$ variety $X = \mathcal{X} \otimes F$ (expln) over F defined / K

[let "script" denote var's / $\mathbb{C}$]

vice versa if $\mathbb{C} \subset F = \overline{F}$, $\forall$ var X/F $\exists$ $\mathcal{X} \to \mathcal{B}$ s.t $X = \mathcal{X} \otimes F$

(L) Assume X proj curve. Can arrange $\mathcal{X} \to \mathcal{B}$ w/ proj fibres &

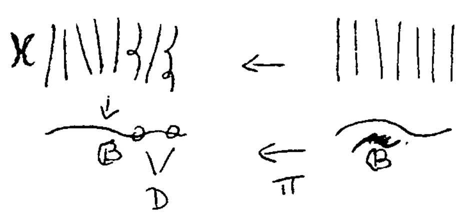
X iso to var def / $\mathbb{C}$ $\Longleftrightarrow$ all fibres of $\mathcal{X} \to \mathcal{B}$ over open set of $\mathcal{B}$ are pairwise isomorphic

fixed affine curve/$\mathbb{C}$ — We $\mathcal{X} \to \mathcal{B}$, $\mathcal{X}_b = \pi^{-1}(b)$ (no assumption on dep of $\mathcal{X}_b$ on b) $\xrightarrow{f_b}$ $\mathcal{X}_b$ fam of proj curves of genus ≥ 2

"well known" Idea "$\Rightarrow$" "clear"

(L) (rigidity) 19-th century

"$\Leftarrow$" (Describe pf using $\mathrm{Isom}(\mathcal{X}/\mathcal{B}, \mathcal{X}'/\mathcal{B}) \to \mathcal{B}$ w/ $\mathcal{X}' = \mathcal{X}_b \times \mathcal{B}$ — uses moduli argument)



$\pi(\tilde{\mathcal{B}}) \subset \mathcal{B} \setminus D$

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DESCENT THEOREM & PROOF OF THM 1

Descent Theorem Let G be an alg gr $/ F$ (F a δ-closed field)

Let $\delta : \mathcal{O}_G \to \mathcal{O}_G$ be a deriv extending that of F. (+)

Let $V \subset G$ subvar (++) which is horizontal for δ (+) & let

$V \to X$ be a non-ct mor to a curve X of genus $\geqslant 2$.

Then X descends to C ($=$ ct field of F.)

Standard reduction: enough to prove (by ~~spreading~~ spreading out $C \mapsto \mathbb{C}$, $F \mapsto$ curve $\mathcal{B}$ over $\mathbb{C}$)

Descent Th' Let $\mathcal{B}$ an affine curve $/ \mathbb{C}$, $\theta : \mathcal{O}(\mathcal{B}) \to \mathcal{O}(\mathcal{B})$ der$/_{\mathbb{C}}$ hence vect. field (Assume $\neq 0$ at all pts)

$\pi : \mathcal{G} \to \mathcal{B}$ mor of var's, $\mathcal{B} \to \mathcal{G}$ section

$\mu : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \to \mathcal{G}$ mor which makes fibres groups

$\tilde{\theta}$ vect field on $\mathcal{G}$ lifting θ s.t. μ horizontal

$\mathcal{V} \subset \mathcal{G}$ closed subvar, horizontal for $\tilde{\theta}$

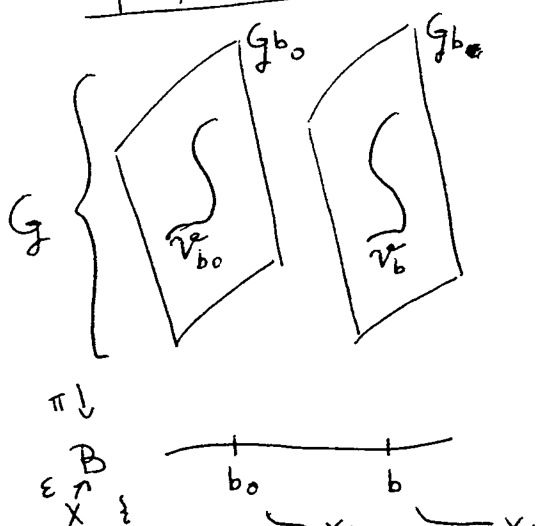
$\varepsilon : \mathbf{X} \to \mathcal{B}$ mor of var w/ fibers curves of genus $\geqslant 2$

$\mathbf{X} \leftarrow \mathcal{V}$ non ct on fibers.

$\searrow_{\mathcal{B}} \swarrow$

Then all fibers of $\mathbf{X} \to \mathcal{B}$ are iso.

Pf of the latter Let $\mathcal{G}_b = \pi^{-1}(b)$, $\mathcal{V}_b = \mathcal{V} \cap \mathcal{G}_b$, $\mathbf{X}_b = \varepsilon^{-1}(b)$



Easy to show $\exists$ analytic (♡) triv (obtained by integrating $\tilde{\theta}$ & using gr. str)

$$\mathcal{G} \times_{\mathcal{B}} \mathcal{D} \underset{an}{\simeq} \mathcal{G}_{b_0} \times \mathcal{D}$$

($\mathcal{D}$ disk in $\mathcal{B}$) s.t. $\tilde{\theta}$ corr to pullback of θ

$\mathcal{V}$ horiz $\Rightarrow$ $\mathcal{V} \times_{\mathcal{B}} \mathcal{D} \underset{an}{\simeq} \mathcal{V}_{b_0} \times \mathcal{D}$

Take curve $\mathcal{W}_{b_0} \subset \mathcal{V}_{b_0}$ not mapped to a pt in $\mathbf{X}_{b_0}$

(In picture $\mathcal{W}_{b_0} = \mathcal{V}_{b_0}$)

* i.e $\delta : \mathcal{O}(U) \to \mathcal{O}(U)$ for all U

** closed!

(++) non-alg!

(+) i.e $\forall$ affine $U \subset X$ if $I =$ ideal of $V \cap U$ in $\mathcal{O}(U)$ then $\delta I \subset I$

♡ see later

Get analytic maps $\mathcal{W}^e_{b_0} \subset \mathcal{V}_{b_0} \underset{\text{analytic}}{\simeq} \mathcal{V}_b \xrightarrow[\text{alg.}]{} X_b$, $b \in \mathcal{D}$

Now Big Picard th says : $\forall$ analytic mor bet an affine alg curve$/\mathbb{C}$ & a proj curve $/\mathbb{C}$, of genus ≥ 2 is alg. (i.e a mor of var)

So $\underbrace{\mathcal{W}^e_{b_0}}_{\substack{\text{fixed}\\ \text{affine curve}}} \to \underbrace{X_b}_{\substack{\text{variable}\\ \text{proj curve}}}$ morph (of varieties)

"Chow rigidity" $\Rightarrow$ all X_b isomorphic to each other. $\square$

<u>Pf of Diff Alg Lang</u> (recall statement p ⑤)

Have $\Sigma = G_*^{sol}$, $G_i \subset J^i(A)$, $G_0 \twoheadleftarrow G_1 \twoheadleftarrow G_2 \twoheadleftarrow \cdots$ connected alg gr.

Since $\sup \dim G_i < \infty$ (& $\mathrm{Ker}(G_{i+1} \to G_i)$ connected) $\Rightarrow$ sequence stabilizes i.e $G_i \leftarrow G_{i+1}$ iso for $i \gg 0$

Call $G = G_i$, $i \gg 0$. Then $\delta : \mathcal{O}_G \circlearrowleft$.

Set $V_i = G_i \cap J^i(X)$. Then $V_i \supset V_{i+1}$ for $i \gg 0$.

By "Noetherian" property $V_i = V_{i+1}$ for $i \gg 0$. Call $V = V_i$ (an irred comp of), $i \gg 0$.

Then $V \subset G$ horizontal. Have $V \subset J^i(X) \to X$.

Hypothesis says X does not desc to C.

Descent Th says $V \to X$ constant.

So for $i \gg 0$, $G_i \cap J^i(X) \to X$ has finite image $\{P_1, \ldots, P_s\}$.

But if $P \in \Sigma \cap X$ then $\nabla_i(P) \in G_i \cap J^i(X)$ so $P \in \{P_1, \ldots, P_s\}$

$\square$

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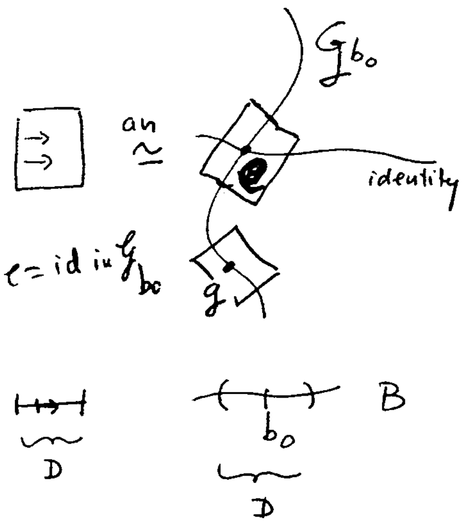
Rem. Let us explain analytic triviality $\mathcal{G} \times_B \mathcal{D} \underset{an}{\simeq} \mathcal{G}_{b_0} \times \mathcal{D}$ (arg of Hamm)
(arg ∈ Hamm in a very general context) ~~May ass~~ ~~Enough to show this for $\mathcal{D}$ a small disk about given pt~~ Assume for simplic $\mathcal{G}_b$ connected



∃ th for ODE ⇒ ∃ disk $\mathcal{D}$ & polydisk E & analytic iso of $\mathcal{D} \times E \underset{an}{\overset{\sigma}{\simeq}}$ open set in $\mathcal{G} / \mathcal{D}$ s.t. pull back of $\theta \longleftrightarrow \tilde{\theta}$. & s.t. $\sigma(b_0 \times E) \ni e$.

Consider

$$\tilde{\mathcal{G}}_{b_0} = \{ g \in \mathcal{G}_{b_0} \mid \exists \text{ polydisk } E \ \& \ \exists\, \tau: \mathcal{D} \times E \underset{an}{\simeq} \text{open in } \mathcal{G}/\mathcal{D} \text{ s.t. p.b. } \theta \leftrightarrow \tilde{\theta} \text{ s.t. } \tau(b_0 \times E) \ni g \}$$

Using fact that $\mu: \mathcal{G} \times_B \mathcal{G} \to \mathcal{G}$ is horiz ⇒ $\tilde{\mathcal{G}}_{b_0}$ open Lie subgr of $\mathcal{G}_{b_0}$.

~~$\mathcal{G}_{b_0}$ has fin many conn. comp's. Take $P_1, \dots, P_N$ in each comp. Set $\mathcal{D} = \mathcal{D}_{P_1} \cap \dots \cap \mathcal{D}_{P_N}$. Get analytic triv / $\mathcal{D}$.~~

$\Rightarrow \tilde{\mathcal{G}}_{b_0} = \mathcal{G}_{b_0}$. $\square$