Dwork's approach to congruence ζ (SKETCH)

$F = \mathbb{F}_q$, $f \in F[x_1, \dots, x_n]$, $X = Z(f)$, $N_s = \# X(\mathbb{F}_{q^s})$, $Z(T) = \exp\left(\sum_{s=1}^{\infty} N_s \frac{T^s}{s}\right) \in \mathbb{Q}[[T]]$.

(T) $Z(T) \in \mathbb{Q}(T)$

(L1) $\exists\ \Theta \in \mathbb{Q}_p(\zeta_p)[[T]]$ conv on the ~~closed~~ a disk of $\mathbb{C}_p$ (of rad > 1) s.t. $\forall a \in \mathbb{F}_{q^s}$ have

$$\zeta_p^{\operatorname{Tr} a} = \Theta(\alpha)\cdot\Theta(\alpha^p)\cdot\Theta(\alpha^{p^2})\cdots\Theta(\alpha^{p^{s-1}}),\ \Big|_{T = \operatorname{Teich}(a)} \quad \alpha = \operatorname{Teich}(a), \text{ where } \operatorname{Teich}: \mathbb{F}_p^a \to \mathbb{C}_p \text{ Teich lift}$$

[Wrong pf would be : $\zeta_p^{\operatorname{Tr} a} \underset{\pi = 1-\zeta_p}{=} (1-\pi)^{a + a^p + \dots + a^{p^{s-1}}} \underset{\alpha = \operatorname{Teich} a}{=} (1-\pi)^{\alpha}(1-\pi)^{\alpha^p}\cdots(1-\pi)^{\alpha^{p^{s-1}}}$ but $(1-\pi)^{\alpha^{p^i}}$ is diverg as a series in α !]

[Right pf see Koblitz]

(L2) Let $G \in R_0 := \{F = \sum f_w x^w \mid f_w \in \mathbb{C}_p$, conv in a disk of rad > 1$\}$ $\circlearrowleft T_q(\sum a_w x^w) = \sum a_{qw} x^w$.

let $\Psi: R_0 \to R_0$, $\Psi(F) = T_q(GF)$.

Then 1) $\operatorname{Tr}(\Psi^s)$ conv for all s & $(q^s-1)\operatorname{Tr}(\Psi^s) = \sum_{\substack{\alpha \in \mathbb{C}_p^n \\ \alpha^{p^s-1} = 1}} G(\alpha)\, G(\alpha^p) \cdots G(\alpha^{p^{s-1}})$

2) $\det(1 - \Psi T) = \exp_p\left\{-\sum_{s=1}^{\infty} \operatorname{Tr}(\Psi^s)\frac{T^s}{s}\right\} \in \mathbb{C}_p[[T]]$ is well def & has ∞ rad of conv

[Note: 2) as in findim case ; 1) is a direct comp.]

Let $N_s' = \#(X(\mathbb{F}_{q^s}) \setminus \text{coord hypplanes})$, $x_0 f(x) = \sum b_w x^w \in F[x_0, x_1, \dots, x_n]$, $\beta_w = \operatorname{Teich}(b_w) \in \mathbb{C}_p$

Then :

1) $\sum_{a_0 \in \mathbb{F}_{q^s}} \zeta_p^{\operatorname{Tr}(a_0 u)} = \begin{cases} 0, & u \in \mathbb{F}_{q^s}^\times \\ q^s, & u = 0 \end{cases}$ $\quad \left(\Leftarrow \sum_{b \in \mathbb{F}_q} \zeta_p^{\operatorname{Tr} b} \overset{\text{EX}}{=} 0\right)$

2) $\sum_{a_0 \in \mathbb{F}_{q^s}^\times} \zeta_p^{\operatorname{Tr}(a_0 u)} = \begin{cases} -1, & u \in \mathbb{F}_{q^s}^\times \\ q^s - 1, & u = 0 \end{cases}$ (by 1) above)

3) $\sum_{a_0, a_1, \dots, a_n \in \mathbb{F}_{q^s}^\times} \zeta_p^{\operatorname{Tr}(a_0 f(a_1, \dots, a_n))} = N_s'(q^s-1) + [(q^s-1)^n - N_s']\cdot(-1) = q^s N_s' - (q^s-1)^n$

(by 2) above)

4) $q^s N_s' = (q^s-1)^n + \uparrow = (q^s-1)^n + \sum_{\substack{\alpha_0, \dots, \alpha_n \in \mathbb{C}_p \\ \alpha_i^{p^s-1} = 1}} \zeta_p^{\sum_w \sum_{i=0}^{s-1} \beta_w^{p^i} \alpha^{w \cdot p^i}}$

$\underset{\text{(L1)}}{=} (q^s-1)^n + \sum_{\substack{\alpha_0, \dots, \alpha_n \in \mathbb{C}_p \\ \alpha_i^{p^s-1} = 1}} \prod_w \Theta(\beta_w \alpha^w)\,\Theta(\beta_w^p \alpha^{pw}) \cdots \Theta(\beta_w^{p^{s-1}} \alpha^{p^{s-1} w})$

$= (q^s - 1)^n + (q^s - 1)\,\mathrm{Tr}(\psi^s)$

$\uparrow$

$G(T) = \prod_w \Theta(\beta_w T)$

& (L2)

5) Define $Z'(T)$ with N' in place of N. Then $Z'(T) = \exp\left(\sum_{s=1}^{\infty} q^{-s}\left[(q^s-1)^n + (q^s-1)\,\mathrm{Tr}(\psi^s)\right]\frac{T^s}{s}\right)$

$=$ expression related to $\det(1 - \psi T)$

6) $Z(T)$ alternate product of $Z'(T)$'s of smaller dim'nl var's

So $Z(T)$ quotient of series w/ ∞ rad of conv

7) $Z(T) \in \mathbb{Q}(T)$ (see Koblitz).