Arithmetic Differential Equations

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2000 *Mathematics Subject Classification.* Primary 14G20, 14G35, 30F10, 14H52 ;
Secondary 53A55, 37F05, 14L24

Key words and phrases. $p-$adic numbers, correspondences, uniformization

ABSTRACT. This research monograph develops an arithmetic analogue of the theory of ordinary differential equations in which derivations are replaced by Fermat quotient operators. The theory is then applied to the construction and study of certain quotient spaces of algebraic curves with respect to correspondences having infinite orbits.

Preface

The main purpose of this research monograph is to develop an arithmetic analogue of the theory of ordinary differential equations. In our arithmetic theory the "time variable" t is replaced by a fixed prime integer p. Smooth real functions, $t \mapsto x(t)$, are replaced by integer numbers $a \in \mathbf{Z}$ or, more generally, by integers in various (completions of) number fields. The derivative operator on functions,

$$x(t) \mapsto \frac{dx}{dt}(t),$$

is replaced by a "Fermat quotient operator" δ which, on integer numbers, acts as $\delta : \mathbf{Z} \to \mathbf{Z}$,

$$a \mapsto \delta a := \frac{a - a^p}{p}.$$

Smooth manifolds (configuration spaces) are replaced by algebraic varieties defined over number fields. Jet spaces (higher order phase spaces) of manifolds are replaced by what can be called "arithmetic jet spaces" which we construct using δ in place of d/dt. Usual differential equations (viewed as functions on usual jet spaces) are replaced by "arithmetic differential equations" (defined as functions on our "arithmetic jet spaces"). Differential equations (Lagrangians) that are invariant under certain group actions on the configuration space are replaced by "arithmetic differential equations" that are invariant under the action of various correspondences on our varieties.

As our main application we will use the above invariant "arithmetic differential equations" to construct new quotient spaces that "do not exist" in algebraic geometry. To explain this we start with the remark that (categorical) quotients of algebraic curves by correspondences that possess infinite orbits reduce to a point in algebraic geometry. In order to address the above basic pathology we propose to "enlarge" algebraic geometry by replacing its algebraic equations with our more general arithmetic differential equations. The resulting new geometry is referred to as *δ−geometry*. It then turns out that certain quotients that reduce to a point in algebraic geometry become interesting objects in δ−geometry; this is because there are more invariant "arithmetic differential equations" than invariant algebraic equations. Here are 3 classes of examples for which this strategy works:

1) *Spherical case.* Quotients of the projective line $\mathbf{P}^1$ by actions of certain finitely generated groups (such as $SL_2(\mathbf{Z})$);

2) *Flat case.* Quotients of $\mathbf{P}^1$ by actions of postcritically finite maps $\mathbf{P}^1 \to \mathbf{P}^1$ with (orbifold) Euler characteristic zero;

3) *Hyperbolic case.* Quotients of modular or Shimura curves (e.g. of $\mathbf{P}^1$) by actions of Hecke correspondences.

Our results will suggest a general conjecture according to which the quotient of a curve (defined over a number field) by a correspondence is non-trivial in $\delta-$geometry for almost all primes p if and only if the correspondence has an "analytic uniformization" over the complex numbers. Then the 3 classes of examples above correspond to *spherical, flat, and hyperbolic* uniformization respectively.

Material included. The present book follows, in the initial stages of its analysis, a series of papers written by the author [**17**]-[**28**]. A substantial part of this book consists, however, of material that has never been published before; this includes our Main Theorems stated at the end of Chapter 2 and proved in the remaining Chapters of the book. The realization that the series of papers [**17**]-[**28**] consists of pieces of one and the same puzzle came relatively late in the story and the unity of the various parts of the theory is not easily grasped from reading the papers themselves; this book is an attempt at providing, among other things, a linear, unitary account of this work. Discussed are also some of the contributions to the theory due to C. Hurlburt [**71**], M. Barcau [**2**], and K. Zimmerman [**29**].

Material omitted. A problem that was left untouched in this book is that of putting together, in an adelic picture, the various $\delta-$geometric pictures, as p varies. This was addressed in our paper [**26**] where such an adelic theory was developed and then applied to providing an arithmetic differential framework for functions of the form $(p,a) \mapsto c(p,a)$, p prime, $a \in \mathbf{Z}$, where $L(a,s) = \sum_n c(n,a)n^{-s}$ are various families of $L-$functions parameterized by a. Another problem not discussed in this book is that of generalizing the theory to higher dimensions. A glimpse into what the theory might look like for higher dimensional varieties can be found in [**17**] and [**3**]. Finally, we have left aside, in this book, some of the Diophantine applications of our theory such as the new proof in [**18**] of the Manin-Mumford conjecture about torsion points on curves and the results in [**21**], [**2**] on congruences between classical modular forms.

Prerequisites. For most of the book, the only prerequisites are the basic facts of algebraic geometry (as found, for instance, in R. Hartshorne's textbook [**66**]) and algebraic number theory (as found, for instance, in Part I of S. Lang's textbook [**87**]). In later Chapters more background will be assumed and appropriate references will be given. In particular the last Chapter will assume some familiarity with the $p-$adic theory of modular and Shimura curves. From a technical point of view the book mainly addresses graduate students and researchers with an interest in algebraic geometry and / or number theory. However, the general theme of the book, its strategy, and its conclusions should appeal to a general mathematical audience.

Plan of the book. We will organize our presentation around the motivating "quotient space" theme. So quotient spaces will take center stage while "arithmetic jet spaces" and the corresponding analogies with the theory of ordinary differential equations will appear as mere tools in our proofs of $\delta-$geometric theorems. Accordingly, the Introduction starts with a general discussion of strategies to construct quotient spaces and continues with a brief outline of our $\delta-$geometric theory. We also include, in our Introduction, a discussion of links, analogies, and / or discrepancies between our theory and a number of other theories such as: differential equations on smooth manifolds [**114**], the Ritt-Kolchin differential algebra [**117**], [**84**], [**32**], [**13**], the difference algebraic work of Hrushovski and Chatzidakis [**69**], the theory of dynamical systems [**109**], Connes' non-commutative geometry [**36**], the theory of Drinfeld modules [**52**], Dwork's theory [**53**], Mochizuchi's $p-$adic Teichmüller theory [**111**], Ihara's theory of congruence relations [**73**] [**74**], and the work of Kurokawa, Soulé, Deninger, Manin, and others on the "field with one element" [**86**], [**128**], [**98**], [**46**]. In Chapter 1 we discuss some algebro-geometric preliminaries; in particular we discuss analytic uniformization of correspondences on algebraic curves. In Chapter 2 we discuss our $\delta-$geometric strategy in detail and we state our main conjectures and a sample of our main results. In Chapters 3, 4, 5 we develop the general theory of arithmetic jet spaces. The corresponding 3 Chapters deal with the global, local, and birational theory respectively. In Chapters 6, 7, 8 we are concerned with our applications of $\delta-$geometry to quotient spaces: the corresponding 3 Chapters are concerned with correspondences admitting a spherical, flat, or hyperbolic analytic uniformization respectively. Details as to the contents of the individual Chapters are given at the beginning of each Chapter. All the definitions of new concepts introduced in the book are numbered and an index of them is included after the bibliography. A list of references to the main results is given at the end of the book. Internal references of the form Theorem $x.y$, Equation $x.y$, etc. refer to Theorems, Equations, etc. belonging to Chapter x (if $x \neq 0$) or the Introduction (if $x = 0$). Theorems, Propositions, Lemmas, Corollaries, Definitions, and Examples are numbered in the same sequence; Equations are numbered in a separate sequence. Here are a few words about the dependence between the various Chapters. The impatient reader can merely skim through Chapter 1; he/she will need to read at least the (numbered) "Definitions" (some of which are not standard). Chapters 2-5 should be read in a sequence. Chapters 6-8 are largely (although not entirely) independent of one another but they depend upon Chapters 2-5.

Acknowledgments. The author wishes to thank D. Bertrand, P. Deligne, E. Hrushovski, Y. Ihara, M. Kim, Y. I. Manin, B. Mazur, S. Lang, A. Pillay, B. Poonen, F. Pop, T. Scanlon, J. Tate, D. Thakur, D. Ulmer, and J. F. Voloch for encouragement and discussions at various stages of development of these ideas. While writing this book the author was supported in part by NSF Grant #0096946.

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Albuquerque, May 2005

Introduction

We start, in Section 1, by explaining our main motivation which comes from the fact that categorical quotients of correspondences on curves tend to be trivial in algebraic geometry. In Section 2 we explain how one can fix this problem for a remarkable class of correspondences; this is done by developing a geometry, called $\delta-$*geometry*, which is obtained from usual algebraic geometry by adjoining a "Fermat quotient operator" δ. If one views δ as an analogue of a derivation with respect to a prime number then $\delta-$geometry can be viewed as obtained from usual algebraic geometry by replacing algebraic equations with "arithmetic differential equations". In Section 3 we discuss relations between our theory and some other theories.

The present Introduction is written in an informal style; a formal presentation of this material will be made in the body of the book.

0.1. Motivation and strategy

0.1.1. Correspondences and their categorical quotients. It is convenient to start in complete generality by considering an arbitrary category $\mathcal{C}$. (Morally $\mathcal{C}$ should be viewed as a category of "spaces" in some geometry.) A *correspondence* in $\mathcal{C}$ is a tuple $\mathbf{X} = (X, \tilde{X}, \sigma_1, \sigma_2)$ where X and $\tilde{X}$ are objects of $\mathcal{C}$ and $\sigma_1, \sigma_2 : \tilde{X} \to X$ are morphisms in $\mathcal{C}$. We sometimes write $\mathbf{X} = (X, \sigma)$, where $\sigma := (\tilde{X}, \sigma_1, \sigma_2)$. Following the standard terminology of geometric invariant theory [**113**] we define a *categorical quotient* for $\mathbf{X}$ to be a pair (Y, π) where Y is an object of $\mathcal{C}$ and $\pi : X \to Y$ is a morphism in $\mathcal{C}$ satisfying the following properties: 1) $\pi \circ \sigma_1 = \pi \circ \sigma_2$; 2) For any pair (Y', π') where Y' is an object of $\mathcal{C}$ and $\pi' : X \to Y'$ is a morphism such that $\pi' \circ \sigma_1 = \pi' \circ \sigma_2$ there exists a unique morphism $\gamma : Y \to Y'$ such that $\gamma \circ \pi = \pi'$. Categorical quotients are sometimes referred to as *co-equalizers*. Of course, if a categorical quotient (Y, π) exists then it is unique up to isomorphism and we shall write $Y = X/\sigma$. Correspondences form, in a natural way, a category: a morphism $\mathbf{X} \to \mathbf{X}'$ between two correspondences $\mathbf{X} = (X, \tilde{X}, \sigma_1, \sigma_2)$ and $\mathbf{X}' = (X', \tilde{X}', \sigma_1', \sigma_2')$ is, by definition, a pair of morphisms $(\pi, \tilde{\pi})$, $\pi : X \to X'$, $\tilde{\pi} : \tilde{X} \to \tilde{X}'$, such that $\pi \circ \sigma_i = \sigma_i' \circ \tilde{\pi}$, $i = 1, 2$. We will assume, for each category $\mathcal{C}$ we shall be considering, that a class of objects in $\mathcal{C}$ is given which we refer to as *trivial* objects. (Morally trivial objects should be viewed as spaces that reduce to a point.)

An important example of correspondences is provided by (discrete) *dynamical systems* (i.e. self maps). Indeed if X^* is an object in our category $\mathcal{C}$ and $s : X^* \to X^*$ is a morphism then one can attach to these data a correspondence $\mathbf{X}^* = (X^*, X^*, id_{X^*}, s)$. More generally if we assume $\mathcal{C}$ possesses fiber products then one can consider, in a natural way, the pull-back $\mathbf{X} := f^*\mathbf{X}^*$ of $\mathbf{X}^*$ via any morphism $f : X \to X^*$; cf. Equation **??**. Categorical quotients of correspondences of the form $\mathbf{X}^*$, or more generally of the form $\mathbf{X}$, should be viewed as (categorical) *spaces of orbits* of the "dynamical system defined by s".

Another example of correspondences appearing in nature are groupoids. To review this we shall tacitly assume that all products intervening (explicitly or implicitly) in the discussion below exist in $\mathcal{C}$. A *groupoid* in $\mathcal{C}$ is a tuple

$$(X, \tilde{X}, \sigma_1, \sigma_2, \nu, \iota, \epsilon)$$

where X and $\tilde{X}$ are objects in $\mathcal{C}$ and

$$\sigma_1 : \tilde{X} \to X, \quad \sigma_2 : \tilde{X} \to X, \quad \nu : \tilde{X} \times_{\sigma_2, X, \sigma_1} \tilde{X} \to \tilde{X}, \quad \iota : \tilde{X} \to \tilde{X}, \quad \epsilon : X \to \tilde{X}$$

are morphisms such that ν satisfies the usual associativity axiom, ϵ satisfies $\sigma_1 \circ \epsilon = \sigma_2 \circ \epsilon = id_X$ plus the usual unit axiom, and ι satisfies $\sigma_1 \circ \iota = \sigma_2$, $\sigma_2 \circ \iota = \sigma_1$ plus the usual inverse axiom. The datum $(X, \tilde{X}, \sigma_1, \sigma_2)$ can then be viewed as a correspondence.

In their turn groupoids often arise from group actions. Again we tacitly assume that all products intervening (explicitly or implicitly) in the discussion below exist in $\mathcal{C}$. Let G be a group object in the category $\mathcal{C}$ and $\mu : G \times X \to X$ an action on an object X in $\mathcal{C}$. Then one can consider the groupoid

$$(X, G \times X, pr_2, \mu, \nu, \iota, \epsilon)$$

where pr_2 is the second projection, and ν, ι, ϵ are naturally induced by the group operations and the action. So in particular group actions $\mu : G \times X \to X$ define correspondences

$$(X, G \times X, pr_2, \mu).$$

Many of the examples of correspondences we shall be interested in will not come, however, from dynamical systems or groupoids.

0.1.2. Basic pathology. Typically it turns out that, in many classical geometric situations, one has interesting correspondences $\mathbf{X} = (X, \sigma) = (X, \tilde{X}, \sigma_1, \sigma_2)$ whose categorical quotient X/σ is trivial. This "basic pathology" manifests itself, for instance, in the case when $\mathcal{C}$ is the category of algebraic varieties over an algebraically closed field and the trivial objects are the points: if X and $\tilde{X}$ above are algebraic varieties and the smallest equivalence relation $\langle \sigma \rangle \subset X \times X$ containing the image of $\sigma_1 \times \sigma_2 : \tilde{X} \to X \times X$ is Zariski dense in $X \times X$ then the categorical quotient X/σ is trivial. Examples of this kind are very common. For instance assume that $dim\ X = dim\ \tilde{X} = 1$, and σ has an *infinite orbit* by which we mean that there exists an infinite sequence of points $P_1, P_2, P_3, ... \in X$ and a sequence of points $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, ... \in \tilde{X}$ such that $\sigma_1(\tilde{P}_i) = P_i$, $\sigma_2(\tilde{P}_i) = P_{i+1}$ for all $i \geq 1$. Then $\langle \sigma \rangle \subset X \times X$ is dense in $X \times X$ and hence X/σ reduces to a point in the category of varieties. Note, by the way, that in the example above it is impossible to embed $\mathbf{X}$ into a correspondence $\mathbf{X}' = (X', \tilde{X}', \sigma_1', \sigma_2')$ in the category of schemes of finite type such that $\mathbf{X}'$ admits a groupoid structure and $dim\ X' = dim\ \tilde{X}' = 1$; here, by an *embedding* we mean a morphism $(\iota, \tilde{\iota}) : \mathbf{X} \to \mathbf{X}'$ with ι and $\tilde{\iota}$ embeddings. Intuitively, $\mathbf{X}$ does not "generate" a groupoid "of finite type" and, in some sense, this is what prevents usual algebraic geometry from "controlling" the quotient.

With the above "basic pathology" in mind one is tempted to "enlarge" usual algebraic geometry so as to make X/σ non-trivial; one can then pursue the study of X/σ in this enlarged geometry. The present book is an attempt to develop such an enlarged geometry and to apply it to the study of categorical quotients.

Before proceeding to explain our approach it might be useful to put things in perspective by discussing some general aspects of quotient theory.

0.1.3. Two viewpoints in quotient theory. Assume we are given a correspondence $\mathbf{X} = (X, \sigma)$ in some category of "spaces" (e.g. smooth manifolds, varieties, schemes, etc.) There are (at least) two possible paths towards the idea of "quotient" of X by σ which we shall call here, for convenience, *invariant theory* and *groupoid theory.*

In invariant theory one seeks to construct a categorical quotient as a "genuine space" X/σ such that the (locally defined) functions on it identify with the (locally defined) "$\sigma-$invariant functions" on X. (Here a locally defined function φ on X is called $\sigma-$*invariant* if $\varphi \circ \sigma_1 = \varphi \circ \sigma_2$ on their common domain of definition). This viewpoint is actually quite successful in algebraic geometry, in case (X, σ) comes from a reductive algebraic group action $\mu : G \times X \to X$; geometric invariant theory codifies the situation in this case. On the other hand the invariant theoretic approach in algebraic geometry utterly fails for correspondences with Zariski dense equivalence relation $\langle \sigma \rangle$; these will be, by the way, the correspondences that will be at the heart of this book.

In groupoid theory one assumes that the correspondence $\mathbf{X} = (X, \tilde{X}, \sigma_1, \sigma_2)$ has a groupoid structure (ν, ι, ϵ) and one *defines* the "space X/σ" to be the groupoid $(X, \tilde{X}, \sigma_1, \sigma_2, \nu, \iota, \epsilon)$ itself regarded up to an appropriate equivalence on the class of groupoids. (This equivalence should allow roughly speaking to transfer modules from one groupoid to another in a sense that depends on the particular context we are in.) Non-commutative geometry and stack theory adopt this viewpoint. By the way, one of Connes' original motivations for developing non-commutative geometry [**36**] was to address the "basic pathology" above as it manifests itself in topology and differential geometry; this has been enormously successful and can be adapted to algebro-geometric situations [**127**], [**99**]. On the other hand passing from algebraic varieties to algebraic stacks, as one sometimes does in moduli space theory, is not sufficiently drastic to make the "basic pathology" go away !

0.1.4. Strategy of the present approach. Our approach towards the "basic pathology" above will use the viewpoint of invariant theory (as opposed to that of groupoid theory); but in order that invariant theory be non-trivial we will have to enlarge algebraic geometry by adjoining to it some new functions. Having "more functions" will increase our chances to find, as we will, interesting $\sigma-$invariant functions. Now there is a general recipe to enlarge algebraic geometry by adjoining to its functions an extra function, δ, satisfying "polynomial compatibility conditions" with respect to addition and multiplication. Cf. [**20**]; see also the last Section of Chapter 2 in our book. Under a certain genericity condition it turns out, as we shall see, that "locally" there are exactly 4 types of such δ's which can be referred to as derivation operators, difference operators, $p-$derivation operators (= "Fermat quotient operators"), and $p-$difference operators. If δ is a derivation one is led to the *differential algebra* (and corresponding geometry) of Ritt [**117**] and Kolchin [**84**]; cf. also [**32**], [**12**], [**14**], [**15**], [**16**]. If δ is a difference operator one is led to the *difference algebra* (and corresponding geometry) of Ritt-Cohn [**35**]; cf. also the work of Chatzidakis-Hrushovski [**33**] and Hrushovski [**70**]. If δ are $p-$derivation operators attached to various prime numbers p one is led to what can be called *arithmetic differential algebra* (and its corresponding geometry). The case when δ is a $p-$difference operator seems to lead to a less interesting theory. Of the above 4 types of δ, $p-$derivations are the only ones that do not vanish identically on the integers. This makes $p-$derivations especially suited for arithmetic applications; so we shall exclusively be interested, in this book, in arithmetic differential algebra. The study of arithmetic differential algebra and its associated geometry was begun in our paper [**17**] and developed in a series of papers [**18**]-[**28**], [**2**], [**3**], [**71**]. Earlier papers (cf. especially papers by Joyal [**79**] and Ihara [**75**]) contain indications that Fermat quotients should be viewed as analogues of derivations. However note that there is an important difference between our approach and the one proposed by Ihara; cf. the remarks surrounding Equation 0.31 below. Arithmetic differential algebra can be viewed as an "arithmetic" analogue of the Ritt-Kolchin differential algebra and also as a "regularized" analogue of the Cohn difference algebra; indeed arithmetic differential algebra can be viewed as obtained from difference algebra by adjoining to it certain divergent series (cf. Remark **??** below).

As a matter of terminology note that, in the Ritt-Kolchin differential algebra one uses the symbol δ as an abbreviation for "differential". By analogy with the tradition in the Ritt-Kolchin theory we will use the symbol δ as an abbreviation for "arithmetic differential"; in particular arithmetic differential algebra will be referred to as $\delta-$*algebra* while its "associated" geometry, which can be called *arithmetic differential geometry*, will be referred to as $\delta-$*geometry*. There will be no danger of confusion with differential algebra and difference algebra terminology because no use of the latter two types of algebra will be made in this book.

The main purpose of this book is to first develop some of the basic elements of $\delta-$geometry and then to construct (and study) interesting categorical quotients, in $\delta-$geometry, for correspondences whose categorical quotient in algebraic geometry is trivial. A conjecture will emerge to the effect that the $\delta-$geometric picture is interesting if and (essentially) only if our correspondences admit a complex analytic uniformization. Below we give a rough outline of our theory; the details of the theory will be explained in Chapter 2.

0.2. Rough outline of the theory

0.2.1. Background and notation. We start with a number field F (always assumed of finite degree over $\mathbf{Q}$). Fix a finite place $\wp$ of F which is unramified over $\mathbf{Q}$; later we will vary $\wp$. We identify $\wp$ with a maximal ideal in the ring of integers $\mathcal{O}_F$ of F. Let $p = char\ \mathcal{O}/\wp$. Let $\mathcal{O}_\wp$ be the localization of $\mathcal{O}_F$ at $\wp$, consider its completion $\hat{\mathcal{O}}_\wp$, and let $\hat{\mathcal{O}}_\wp^{ur}$ be the maximum unramified extension of $\hat{\mathcal{O}}_\wp$ (obtained by adjoining to $\hat{\mathcal{O}}_\wp$ all the roots of unity of order prime to p in an algebraic closure of its fraction field). Then $\hat{\mathcal{O}}_\wp^{ur}$ is a discrete valuation ring with maximal ideal generated by p. Consider the completion of $\hat{\mathcal{O}}_\wp^{ur}$ which we denote by

$$R_\wp := (\hat{\mathcal{O}}_\wp^{ur})\hat{}.$$

The ring $R_\wp$ has a simple well known structure: any element of it can be represented uniquely as a series $\sum_{i=0}^\infty \zeta_i p^i$ where ζ_i are roots of unity of order prime to p or 0. The ring $R_\wp$ has a unique automorphism ϕ inducing the $p-$power Frobenius map on the residue field $k_\wp := R_\wp/pR_\wp$; it is given by $\phi(\sum_{i=0}^\infty \zeta_i p^i) = \sum_{i=0}^\infty \zeta_i^p p^i$. For

$F = \mathbf{Q}$, $\wp = (p)$, we have $\hat{\mathbf{Z}}_{(p)} = \mathbf{Z}_p$, the ring of $p-$adic numbers, so we simply write $R_p = \hat{\mathbf{Z}}_p^{ur}$ in place of $(\hat{\mathbf{Z}}_{(p)}^{ur})\hat{}$. By the way, for a general F and $\wp$ the natural morphism $R_p \to R_\wp$ is an isomorphism so the ring $R_\wp$ depends only on p and not on F and $\wp$.

0.2.2. $\delta-$**geometry.** With the notation above we let $\delta = \delta_p : R_\wp \to R_\wp$ be the "Fermat quotient operator" defined by $\delta x = (\phi(x) - x^p)/p$. We consider a "geometric category", $\mathcal{C}_\delta$, which morally underlies $\delta-$*geometry at* $\wp$. The objects of $\mathcal{C}_\delta$ will be called $\delta-$*ringed sets*; a $\delta-$ringed set X_δ consists of an underlying set X_{set} equipped with a family (X_s) of subsets and a family of "structure rings" $(\mathcal{O}_s)$ of $R_\wp-$valued functions on the X_s's such that $f \in \mathcal{O}_s$ implies $\delta \circ f \in \mathcal{O}_s$. Here s runs through a monoid S and one assumes that $X_{st} = X_s \cap X_t$ for $s, t \in S$. One also assumes compatibility with restrictions i.e. $f \in \mathcal{O}_s$ implies $f_{|X_{st}} \in \mathcal{O}_{st}$. A $\delta-$ringed set is called *trivial* if $\mathcal{O}_s = R_\wp$ for all s. There is a natural concept of morphism of $\delta-$ringed sets; with these morphisms the $\delta-$ringed sets form a category $\mathcal{C}_\delta$. Any correspondence in $\mathcal{C}_\delta$ has a categorical quotient. So the existence of categorical quotients is not here the issue; the issue will always be their non-triviality.

0.2.3. Passing from algebraic geometry to $\delta-$**geometry.** Let $X_\wp = X$ be a smooth scheme of finite type over $R_\wp$ with irreducible geometric fibers. We will attach to the scheme X a $\delta-$ringed set X_δ; the underlying set X_{set} of X_δ will be the set $X(R_\wp)$ of $R_\wp-$points of X and the structure rings $\mathcal{O}_s$ will consist of certain functions $P \mapsto \varphi(P)$ that locally, in coordinates $x \in R_\wp^d$, look like

$$\varphi(x) = \frac{F(x, \delta x, ..., \delta^r x)}{G(x, \delta x, ..., \delta^r x)}$$

where F, G are restricted power series with $R_\wp-$coefficients. Let us make this precise. First, a function $f : X(R_\wp) \to R_\wp$ is called a $\delta-$*function of order* $\leq r$ if for any $P \in X(R_\wp)$ there is a Zariski open set $U \subset X$, $P \in U(R_\wp)$, and a closed embedding $U \subset \mathbf{A}^d$ such that $f_{|U(R_\wp)}$ is given in coordinates $x \in R_\wp^d$ by

$$f(x) = F(x, \delta x, ..., \delta^r x)$$

where F is a restricted power series with coefficients in $R_\wp$. Recall that a power series is called *restricted* if its coefficients converge $p-$adically to 0. Denote by $\mathcal{O}^r(V)$ the rings of $\delta-$functions of order $\leq r$ on a Zariski open set $V \subset X$. Then $V \mapsto \mathcal{O}^r(V)$ defines a sheaf $\mathcal{O}^r$ of rings on X for the Zariski topology. Define a $\delta-$line bundle to be a locally free sheaf of $\mathcal{O}^r-$ modules of rank one. Consider the ring $W := \mathbf{Z}[\phi] \subset End(R_\wp)$ and let W_+ be the set of all $w = \sum a_i \phi^i \in W$ with $a_i \geq 0$. The multiplicative monoid W acts on $R^\times$ by $\lambda^w = \prod (\phi^i(\lambda))^{a_i}$, $\lambda \in R^\times$. For any $\delta-$line bundle L one can define a $\delta-$line bundle L^w by acting with w upon the defining cocycle of L. Let $L = K^{-1}$ be now the anticanonical bundle on X, viewed as a $\delta-$line bundle. We consider the graded ring

$$\bigoplus_{w \in W_+} H^0(X, L^w).$$

Its homogeneous elements will be called $\delta-$*sections*. We let S be the monoid of $\delta-$sections of weight $\neq 0$, not divisible by p, in the above ring. For any $s \in S$ of degree w_0 we let

$$X_s := \{P \in X(R_\wp) \mid s(P) \not\equiv 0 \ \ mod \ \ p\},$$

$$\mathcal{O}_s := \{P \mapsto \varphi(P) := t(P)/s^w(P) \mid t \in H^0(X, L^{ww_0})\}.$$

The data (X_s), $(\mathcal{O}_s)$ define a $\delta-$ringed set X_δ with underlying set $X(R_\wp)$.

Let now $\mathbf{X}_\wp = \mathbf{X} = (X, \sigma) = (X, \tilde{X}, \sigma_1, \sigma_2)$ be a correspondence in the category of schemes over $R_\wp$. Assume X and $\tilde{X}$ are smooth of finite type over $R_\wp$ and have irreducible geometric fibers. Then by the discussion above one can attach to the schemes X and $\tilde{X}$ two $\delta-$ringed sets X_δ and $\tilde{X}_\delta$. Assume moreover that σ_1 and σ_2 are étale. Then the morphisms $\sigma_1, \sigma_2 : \tilde{X} \to X$ induce morphisms between the corresponding $\delta-$ringed sets. So we end up with a correspondence $\mathbf{X}_\delta = (X_\delta, \sigma_\delta)$ in the category $\mathcal{C}_\delta$. One can easily describe the categorical quotient X_δ/σ_δ in $\mathcal{C}_\delta$. The underlying set of the categorical quotient X_δ/σ_δ is the set of equivalence classes of points in $X(R_\wp)$ with respect to the smallest equivalence relation containing the image of the map

$$\sigma_1 \times \sigma_2 : \tilde{X}(R_\wp) \to X(R_\wp) \times X(R_\wp).$$

Let $\tilde{L}$ be the anticanonical bundle on $\tilde{X}$. Let us say that a $\delta-$section $s \in H^0(X, L^w)$ is $\sigma-$*invariant* if its pull-backs to $H^0(\tilde{X}, \tilde{L}^w)$ via σ_1^* and σ_2^* coincide; $\sigma-$ invariant $\delta-$sections will simply be called $\delta-$ *invariants*. Then the monoid parameterizing the structure rings of X_δ/σ_δ is the monoid of all $\delta-$invariants $s \in H^0(X, L^w)$ which are not divisible by p. For s of degree w_0 the corresponding structure ring of X_δ/σ_δ is

$$\{[P] \mapsto \varphi(P) := t(P)/s^w(P) \mid t \in H^0(X, L^{ww_0}) \quad \text{a } \delta\text{−invariant}\}.$$

So morally X_δ/σ_σ is non-trivial if one can find at least two $R_\wp-$linearly independent $\delta-$invariants in one of the spaces $H^0(X, L^w)$, where $0 \neq w \in W_+$. This agrees, of course, with the ideology of classical invariant theory.

0.2.4. Conjectural picture. At this point we consider a global situation. Let S be a finite set of finite places of the number field F, containing all the places ramified over $\mathbf{Q}$, assume (for simplicity) that S consists of all the places containing some integer $m \in \mathbf{Z}$, and let $\mathcal{O} := \mathcal{O}_F[1/m]$ be the ring of $S-$integers of F. Consider a correspondence

$$\mathbf{X} = \mathbf{X}_\mathcal{O} = (X, \sigma),$$

$\sigma = (\tilde{X}, \sigma_1, \sigma_2)$, in the category $\mathcal{C}_\mathcal{O}$ of schemes over $\mathcal{O}$. We will assume X and $\tilde{X}$ are of finite type and smooth over $\mathcal{O}$, have irreducible geometric fibers of dimension one (so they are "curves" over $\mathcal{O}$), and σ_1, σ_2 are étale. We have induced correspondences

$$\mathbf{X}_\wp = (X_\wp, \sigma_\wp), \quad \mathbf{X}_\mathbf{C} = (X_\mathbf{C}, \sigma_\mathbf{C})$$

in the category of schemes over $R_\wp$ and $\mathbf{C}$ respectively. We will always assume that the smallest equivalence relation in $X_\mathbf{C} \times X_\mathbf{C}$ containing the image of $\tilde{X}_\mathbf{C} \to X_\mathbf{C} \times X_\mathbf{C}$ is Zariski dense in $X_\mathbf{C} \times X_\mathbf{C}$ so that the categorical quotient $X_\mathbf{C}/\sigma_\mathbf{C}$ is trivial in the category of schemes over $\mathbf{C}$. As we already saw, for all $\wp \notin S$, we may consider a correspondence

$$\mathbf{X}_\delta = (X_\delta, \sigma_\delta)$$

in the category $\mathcal{C}_\delta$. Our general guiding conjecture will (roughly speaking) assert that:

The categorical quotient X_δ/σ_δ is non-trivial for almost all places $\wp$ if and (essentially) only if the correspondence $\mathbf{X}_\mathbf{C}$ admits an "analytic uniformization".

Here, by $\mathbf{X_C}$ admitting an *analytic uniformization*, we understand that there exist morphisms of correspondences in the category of (open) Riemann surfaces (i.e. in the category of complex analytic manifolds of dimension one), $(\iota, \tilde{\iota}) : \mathbf{X_C} \to \mathbf{Y_C}$, $(\pi, \tilde{\pi}) : \mathbf{Y'_C} \to \mathbf{Y_C}$, where ι, $\tilde{\iota}$ are open immersions, π, $\tilde{\pi}$ are (possibly infinite) Galois covers unramified above X, $\tilde{X}$, and $\mathbf{Y'_C} = (\mathbf{S}, \mathbf{S}, \tau_1, \tau_2)$, with $\mathbf{S}$ a simply connected Riemann surface and τ_1, τ_2 automorphisms of $\mathbf{S}$. We always assume here that the Galois groups of π and $\tilde{\pi}$ are *admissible* in the sense that they act properly discontinuously on $\mathbf{S}$ and are either finite (in case $\mathbf{S} = \mathbf{P}^1$) or infinite (in case $\mathbf{S} = \mathbf{C}$) or of finite covolume (if $\mathbf{S} = \mathbf{H}$, the upper half plane) respectively. According to whether $\mathbf{S}$ is $\mathbf{P}^1$, $\mathbf{C}$, or $\mathbf{H}$ there are three classes of correspondences admitting analytic uniformizations which will be called *spherical, flat, and hyperbolic* correspondences. Analytically uniformizable correspondences can be completely classified in algebraic terms; cf. Chapter 1. Up to an appropriate equivalence relation we have, very roughly speaking, the following description. (N.B. The actual situation is slightly more complicated.) The spherical uniformizable correspondences are, essentially, of the form $(\mathbf{P}^1, \mathbf{P}^1, \pi, \pi \circ \tau)$ where $\pi : \mathbf{P}^1 \to \mathbf{P}^1$ is a finite Galois cover and $\tau : \mathbf{P}^1 \to \mathbf{P}^1$ is an isomorphism; the categorical quotient in the category of sets of such a correspondence is the same as the quotient of $\mathbf{P}^1$ by the action of the group $\langle \Gamma, \tau \rangle$ generated by τ and the Galois group, Γ, of π. Of course the group $\langle \Gamma, \tau \rangle$ is generally infinite. Flat uniformizable correspondences are, essentially, of the form $(\mathbf{P}^1, \mathbf{P}^1, \sigma_1, \sigma_2)$ where σ_i are either multiplicative functions $t \mapsto t^{d_i}$ or Chebyshev polynomials, or Lattès functions [**109**] (the latter being induced by endomorphisms of elliptic curves E via isomorphisms $E/\langle \gamma \rangle \simeq \mathbf{P}^1$ where $\gamma : E \to E$ is an automorphism). The hyperbolic uniformizable correspondences (with infinite orbits) are, essentially, Hecke correspondences on modular or Shimura curves (that are classically described in terms of quaternion algebras over totally real fields). So the "if" part of our conjecture essentially says that our $\delta-$geometric theory gives a rich picture in all the above examples; the "only if" part of the conjecture says that the above examples are, essentially, the *only* ones for which our $\delta-$geometric theory gives a rich picture.

0.2.5. Results. Our main results show that the "if" part of the conjecture above holds (under some mild assumptions) in the spherical case, in the flat case, and in the "rational hyperbolic" case. (Here the "rational hyperbolic" case refers to the case when $\mathbf{S} = \mathbf{H}$ and the quaternion algebra describing the situation has center $\mathbf{Q}$.) For correspondences $\mathbf{X}$ admitting such a uniformization we will study the "geometry" and "cohomology" of the quotients X_δ/σ_δ. In particular we will show that the quotients X_δ/σ_δ tend to behave like "rational varieties" and the quotient maps $X_\delta \to X_\delta/\sigma_\delta$ tend to look like "pro-finite covers" whose Galois properties we shall study. The "only if" part of the conjecture is much more mysterious. We will be able to prove a local analogue of the "only if" part of our conjecture. Also we will prove a global result along the "only if" direction saying (roughly speaking) that if X_δ/σ_δ is "sufficiently non-trivial" for almost all places $\wp$ then $\mathbf{X_C}$ is *critically finite* (in the sense of complex dynamics) and $\mathbf{X}_\wp \otimes k_\wp$ has a generically trivial *pluricanonical bundle*. The latter property will allow us to prove a version of the "only if" part of the conjecture in the "dynamical system case".

0.2.6. Proofs. Here are a few words about the strategy of our proofs. The first move is to attach to any smooth scheme of finite type $X_\wp$ over $R_\wp = \hat{\mathbf{Z}}_p^{ur}$ a projective system of formal schemes

$$(0.1) \qquad ... \to J^r(X_\wp) \to J^{r-1}(X_\wp) \to ... \to J^0(X_\wp) = \hat{X}_\wp$$

such that for each r the ring $\mathcal{O}^r(X_\wp)$ of $\delta-$functions of order $\leq r$ on $X_\wp(R_\wp)$ identifies with the ring $\mathcal{O}(J^r(X_\wp))$ of global functions on $J^r(X_\wp)$:

$$(0.2) \qquad \mathcal{O}^r(X_\wp) \simeq \mathcal{O}(J^r(X_\wp)).$$

The formal scheme $J^r(X_\wp)$ should be viewed intuitively as an "arithmetic jet space" and will be referred to as the $p-$*jet space* of order r of $X_\wp$. It is an arithmetic analogue of the jet space of a manifold (relative to a submersion) in differential geometry (cf. Equation 0.6) and, in particular, it carries certain structures reminiscent of structures in classical mechanics. The elements of the rings in Equation 0.2 can then be viewed intuitively as "arithmetic differential equations" in the same way in which functions on jet spaces in differential geometry are interpreted as "differential equations". The above construction reduces the study of "$\delta-$geometry" of $X_\wp$ to the study of (usual) algebraic geometry of $J^r(X_\wp)$. Actually the construction of $p-$jet spaces is quite easy to explain. Assume, for simplicity, that $X_\wp$ is affine given by

$$(0.3) \qquad X_\wp = Spec\ R_\wp[T]/(F),$$

where T is a tuple of indeterminates and F is a tuple of elements in $R_\wp[T]$. Assume furthermore, for simplicity, that the components of F have coefficients in $\mathbf{Z}_p$. Then $J^r(X_\wp)$ is, by definition, the formal spectrum

$$(0.4) \qquad J^r(X_\wp) := Spf\ R_\wp[T, T', T'', ..., T^{(r)}]\hat{}/(F, \delta F, \delta^2 F, ..., \delta^r F)$$

where $\hat{}$ means "$p-$adic completion", $T', T'', ...$ are new tuples of variables and $\delta F, \delta^2 F, ...$ are defined by the formulae

$$(0.5) \qquad (\delta F)(T, T') := \frac{F(T^p + pT') - F(T)^p}{p},$$

$$(\delta^2 F)(T, T', T'') := \frac{(\delta F)(T^p + pT', (T')^p + pT'') - ((\delta F)(T, T'))^p}{p},$$

etc. The polynomials $\delta F, \delta^2 F, ...$ should be viewed as arithmetic analogues of "iterated total derivatives" of F and the construction of our "arithmetic jet spaces" is then analogous to that of "differential algebraic jet spaces" to be discussed presently; cf. Equation 0.20 below. Also, like in the case of the latter, and as in the case of differential geometry, the fibers of the maps $J^r(X_\wp) \to J^{r-1}(X_\wp)$ are ($p-$adic completions of) affine spaces.

Having reduced our problems to problems about $p-$jet spaces the next step is to construct, in each of the spherical, flat, and hyperbolic cases, some remarkable $\delta-$*invariants*. In the spherical case this will be elementary. In the flat case the construction will be essentially based on our theory of $\delta-$*characters* [**17**]; the latter are arithmetic analogues of the Manin maps in [**97**]. In the hyperbolic case the construction will be based on our theory of *isogeny covariant* $\delta-$*modular forms* [**21**], [**23**], [**24**]. We will develop the theories of $\delta-$characters and $\delta-$modular forms ab initio in this book; we will partly follow the above cited papers and then we will further develop these theories up to a point where we can use them for our applications.

Finally we will need methods to prove that *all* $\delta-$invariants essentially occur via the constructions mentioned above. There will be a number of methods used for this purpose. There is, for instance, a Galois theoretic method that is sufficient to handle, in particular, the spherical case. Another method, especially useful in the flat and hyperbolic case, is close in spirit to that used by Hilbert in his original approach to invariant theory and is based on the existence of a certain system of (usual) partial differential operators acting on appropriate rings of invariants. In our case the partial differential operators in question will live on arithmetic jet spaces and will be derived by analogy with classical mechanics.

0.3. Comparison with other theories

0.3.1. Differential equations on smooth manifolds. Many of the $\delta-$ geometric concepts to be introduced and studied in this book are arithmetic analogues of concepts related to differential equations on smooth manifolds derived from classical mechanics. Let us quickly review here some of these smooth manifold concepts.

Let M be a smooth manifold (i.e. a C^∞ real manifold) equipped with a submersion $\pi : M \to \mathbf{R}$. For each $t \in \mathbf{R}$ we think of $\pi^{-1}(t)$ as the "configuration space" at time t. (The map $M \to \mathbf{R}$ should be viewed as the object whose arithmetic analogues are the morphisms of schemes, $X \to Spec\ F$, defining varieties X over number fields F.) For any smooth map $f : \mathbf{R} \to M$ we denote by $J_0^r(f)$ the $r-$jet of f at 0. We define the *jet space of M (relative to π)* by

$$J^r(M) := \{J_0^r(f)\ |f : \mathbf{R} \to M \text{ smooth},\ \pi \circ f : \mathbf{R} \to \mathbf{R} \text{ a translation}\}. \tag{0.6}$$

Then $J^r(M)$ has a natural structure of smooth manifold. Any smooth section $s : \mathbf{R} \to M$ of $\pi : M \to \mathbf{R}$ lifts to a smooth map $\mathbf{pr}\ s : \mathbf{R} \to J^r(M)$ defined by

$$(\mathbf{pr}\ s)(t) := J_0^r(s \circ \tau_t),$$

where $\tau_t : \mathbf{R} \to \mathbf{R}$ is the translation $\tau_t(u) := u + t$. The smooth functions $L : J^r(M) \to \mathbf{R}$ can be thought of as (time dependent) *Lagrangians*. Alternatively we may think of such smooth functions L as *differential equations on M*. The natural projections $J^r(M) \to J^{r-1}(M)$ are fiber bundles with fiber $\mathbf{R}^n$, where $dim\ M = n + 1$.

Next, in order to simplify our discussion, we will work in coordinates. So we let $M := \mathbf{R} \times \mathbf{R}^n$ with coordinates $(t, x) = (t, x_1, ..., x_n)$ and we let $\pi : M \to \mathbf{R}$ be the first projection. Then we have a natural identification $J^r(M) = \mathbf{R} \times \mathbf{R}^{n(r+1)}$ sending the $r-$jet $J_0^r(f)$ of a map $f : \mathbf{R} \to M$, $f(t) = (t + t_0, x(t))$, into the tuple

$$\left(t_0, x(0), \frac{dx}{dt}(0), ..., \frac{d^r x}{dt^r}(0)\right).$$

If $s : \mathbf{R} \to \mathbf{R} \times \mathbf{R}^n$, $s(t) = (t, x(t))$, is a section of π then $\mathbf{pr}\ s : \mathbf{R} \to \mathbf{R} \times \mathbf{R}^{n(r+1)}$ is given by

$$(\mathbf{pr}\ s)(t) = \left(t, x(t), \frac{dx}{dt}(t), ..., \frac{d^r x}{dt^r}(t)\right). \tag{0.7}$$

Cf. [**114**], p. 96. We denote by $(t, x, x', ..., x^{(r)})$ the coordinates on $\mathbf{R} \times \mathbf{R}^{n(r+1)}$, where $x', ..., x^{(r)}$ are $n-$tuples of variables; for each index i the variables $x_i', x_i'', ...$ are the variables *conjugate* to x_i. A (time dependent) Lagrangian is simply a smooth function

$$L(t, x, x', ..., x^{(r)}) \tag{0.8}$$

on $\mathbf{R} \times \mathbf{R}^{n(r+1)}$. Below is a list of key concepts related to the above context.

A (vertical) *vector field* on M is a derivation operator (on smooth functions of t, x) of the form

$$V := \sum_i a_i(t,x) \frac{\partial}{\partial x_i} \tag{0.9}$$

where a_i are smooth functions. More generally a (vertical) *generalized vector field* (sometimes called *Bäcklund vector field*) is a derivation operator (on smooth functions of t, x) of the form

$$V := \sum_i a_i(t, x, x', ..., x^{(r)}) \frac{\partial}{\partial x_i}; \tag{0.10}$$

cf. [**114**], p. 289.

The *total derivative* operator is the derivation operator (on smooth functions of $t, x, x', ..., x^{(r)}$) defined by

$$D_t := \frac{\partial}{\partial t} + \sum_{ij} x_i^{(j+1)} \cdot \frac{\partial}{\partial x_i^{(j)}}; \tag{0.11}$$

cf. [**114**], p. 109. It is characterized by the property that

$$(D_t L) \circ (\mathbf{pr}\ s) = \frac{d}{dt}(L \circ \mathbf{pr}\ s)$$

for all Lagrangians L and all sections s of π. One should view D_t as the "generator of the Cartan distribution".

The *prolongation* of a (vertical) vector field as in Equation 0.9 is defined by

$$\mathbf{pr}\ V := \sum_{ij} (D_t^j a_i) \cdot \frac{\partial}{\partial x_i^{(j)}}; \tag{0.12}$$

cf. [**114**], p. 110; it is the unique (smooth) vector field on $J^r(M)$ that commutes with D_t and coincides with V on functions of t and x.

A vector field as in Equation 0.9 is an *infinitesimal symmetry* for an $\mathbf{R}-$linear space $\mathcal{L}$ of Lagrangians if

$$(\mathbf{pr}\ V)(\mathcal{L}) \subset \mathcal{L}. \tag{0.13}$$

If (L_i) is a family of Lagrangians and $\mathcal{L}$ is the ideal generated by the family $(D_t^j L_i)$ in the ring of all Lagrangians then any infinitesimal symmetry of $\mathcal{L}$ in the sense above is an infinitesimal symmetry of the "system of differential equations $L_i = 0$" in the sense of [**114**], p. 161.

The vector field V is a *variational infinitesimal symmetry* of the Lagrangian L if

$$(\mathbf{pr}\ V)(L) = 0; \tag{0.14}$$

cf. [**114**], p. 253.

The *Fréchet derivative* of a Lagrangian L is the operator on vector fields given by

$$V = \sum_i a_i \frac{\partial}{\partial x_i} \mapsto (\mathbf{pr}\ V)(L) = \langle \mathbf{pr}\ V, dL \rangle = \sum_{ij} (D_t^j a_i) \cdot \frac{\partial L}{\partial x_i^{(j)}}; \tag{0.15}$$

cf. [**114**], p. 307.

The *Euler-Lagrange form* attached to a Lagrangian L is the $1-$form

$$\mathbf{EL}(L) := \sum_i \left(\sum_j (-1)^j D_t^j \left(\frac{\partial L}{\partial x_i^{(j)}} \right) \right) dx_i; \tag{0.16}$$

cf. [**114**], p. 246.

Noether's Theorem in this context is the (trivial) statement that for any Lagrangian $L(t, x, x', ..., x^{(r)})$ and any vector field V as in Equation 0.9 there exists $G = G(t, x, x', ..., x^{(2r-1)})$ such that

$$\langle V, \mathbf{EL}(L) \rangle - (\mathbf{pr}\ V)(L) = D_t G. \tag{0.17}$$

If V is a variational infinitesimal symmetry of L (i.e. Equation 0.14 holds) then G is called the *conservation law* attached to V; it is unique up to an additive constant and it has the property that if the function $t \mapsto x(t)$ satisfies the *Euler-Lagrange equations*

$$\langle \frac{\partial}{\partial x_i}, \mathbf{EL}(L) \rangle \left(t, x(t), \frac{dx}{dt}(t), ..., \frac{d^{2r}x}{dt^{2r}}(t) \right) = 0 \tag{0.18}$$

for $i = 1, ..., n$ then the function

$$t \mapsto G \left(t, x(t), \frac{dx}{dt}(t), ..., \frac{d^{2r-1}x}{dt^{2r-1}}(t) \right)$$

is constant.

Finally the *Hamiltonian vector field* attached to a function $H = H(t, x, x')$ and the vector fields $\partial/\partial x_i$ is the vector field

$$\sum_i \left(\frac{\partial H}{\partial x_i'} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial}{\partial x_i'} \right). \tag{0.19}$$

All the concepts above have arithmetic analogues some of which will play a key role in the proofs of our main theorems. The arithmetic analogues of jet spaces of manifolds, of total derivatives, and of Lagrangians were already encountered in our discussion of "arithmetic jet spaces". The arithmetic analogue of prolongation of vector fields (and certain related operators) will play a key role in controlling $\delta-$invariants in our theory. The arithmetic analogues of Fréchet derivatives will be the tangent maps in $\delta-$geometry. In some of our basic $\delta-$geometric examples we will prove the existence of "arithmetic (variational) infinitesimal symmetries", we will discover remarkable systems of commuting "arithmetic Hamiltonian vector fields", and we will compute "arithmetic Euler-Lagrange equations for free particles". The way to the latter is as follows. The Lagrangian of a free particle with one degree of freedom is $(x')^2$; one should view this as attached to the group $\mathbf{G}_a(\mathbf{R}) = (\mathbf{R}, +)$. If one formally sets $y = e^x$ then $(x')^2 = (y'/y)^2$; so $(y'/y)^2$ should be viewed as the Lagrangian for a free particle on the group $\mathbf{G}_m(\mathbf{R}) = (\mathbf{R}^\times, \cdot)$. Now $y \mapsto y'/y$ is the basic *differential character* of $\mathbf{G}_m$ i.e. it is a group homomorphism $(y_1 y_2 \mapsto (y_1'/y_1)+(y_2'/y_2))$ defined by rational functions in the coordinates and their derivatives. Manin maps (cf. the discussion below) provide differential characters for elliptic curves E. On the other hand our theory provides arithmetic analogues, ψ, of differential characters for one dimensional algebraic groups $\mathbf{G}_a$, $\mathbf{G}_m$, E (which we call $\delta-$characters). So it is reasonable to define Lagrangians of free particles (with one degree of freedom) as squares, ψ^2, of $\delta-$characters for one dimensional algebraic groups. Finally our arithmetic theory views $\delta-$characters as "flat" and provides "spherical" and "hyperbolic" analogues of them called, in our book, f^1. Consequently we shall view the squares $(f^1)^2$ as spherical resp. hyperbolic analogues of Lagrangians of free particles (with one degree of freedom).

In order to clarify the analogy between the usual differential equations and arithmetic differential equations it will be useful to discuss an intermediate step which is provided by the Ritt-Kolchin differential algebra [**117**], [**84**].

0.3.2. Differential algebra. The presentation here follows [**13**]. In differential algebra one usually starts with a field K equipped with a derivation $\delta : K \to K$; recall that, by definition, δ is an additive map satisfying the usual Leibniz rule. We shall assume, for simplicity, that K has characteristic zero and is algebraically closed. In applications K usually contains the field $\mathbf{C}(t)$ of rational functions in one variable and δ extends the derivation d/dt. (For the "most general" case of this theory, in which K is replaced by any ring and δ is replaced by a "Hasse-Schmidt derivation" we refer to Vojta's preprint [**131**].) For any non-singular variety X over K one can then define a projective system of non-singular varieties

$$... \to J^r(X) \to J^{r-1}(X) \to ... \to J^0(X) = X$$

called the (differential algebraic) *jet spaces* of X. It is easy to review the construction of these varieties. Assume, for simplicity, that X is affine given by

$$X = Spec\ K[T]/(F)$$

where T is a tuple of indeterminates $T_1, ..., T_N$ and F is a tuple of elements in $K[T]$. Then $J^r(X)$ is, by definition, the spectrum

$$J^r(X) := Spec\ K[T, T', T'', ..., T^{(r)}]/(F, \delta F, \delta^2 F, ..., \delta^r F) \tag{0.20}$$

where $T', T'', ...$ are new tuples of variables and $\delta F, \delta^2 F, ...$ are defined by the formulae

$$(\delta F)(T, T') := F^{\delta}(T) + \sum_j \frac{\partial F}{\partial T_j}(T) \cdot T'_j, \tag{0.21}$$

$$(\delta^2 F)(T, T', T'') := (\delta F)^{\delta} + \sum_j \frac{\partial \delta F}{\partial T_j}(T, T') \cdot T'_j + \sum_j \frac{\partial \delta F}{\partial T'_j}(T, T') \cdot T''_j,$$

etc., where F^{δ}, $(\delta F)^{\delta}$,... are obtained from $F, \delta F, ...$ by applying δ to the coefficients. The polynomials $\delta F, \delta^2 F, ...$ are the *iterated total derivatives* of F. There is a clear analogy between the arithmetic jet spaces in Equation 0.4 and the differential algebraic jet spaces $J^r(X)$ in Equation 0.20. On the other hand it can be shown ([**14**], p.1443) that the $K-$points of the differential algebraic jet space $J^r(X)$ can be identified with the morphisms of $K-$schemes,

$$Spec\ K[[T]]_{e^{T\delta}}/(T^{r+1}) \to X, \tag{0.22}$$

where $K[[T]]_{e^{T\delta}}$ is the ring of power series $K[[T]]$ viewed as a $K-$algebra via the map

$$e^{T\delta} : K \to K[[T]], \quad e^{T\delta}(x) := \sum_{n=0}^{\infty} \frac{\delta^n x}{n!} T^n.$$

(The maps in Equation 0.22 should be viewed as a twisted version of *arcs* in X where the twist is given by δ. Recall that usual *arcs* of order r in X are defined as morphisms of $K-$schemes, $Spec\ K[[T]]/(T^{r+1}) \to X$, where $K[[T]]$ is viewed as a $K-$algebra via the inclusion $K \subset K[[T]]$; the set of all arcs of order r in X has a natural structure of algebraic variety over K and is referred to as the *arc space of* X of order r. Cf. [**45**]. Therefore the varieties $J^r(X)$ can be viewed as a twisted analogue of arc spaces. They identify with arc spaces in case X is defined over the field $C = \{a \in K | \delta a = 0\}$ of constants of δ. But for X not descending to C arc spaces are different from the varieties $J^r(X)$; cf. [**14**] for more on this.) Now there is a clear analogy between the maps in Equation 0.22 and the jets belonging to $J^r(M)$ in Equation 0.6. This makes differential algebraic jet spaces an analogue of jet spaces of smooth manifolds (relative to a submersion). The analogies described above can be summarized in the following diagram:

$$(1) - - - (2) - - - (3), \tag{0.23}$$

(1)=jet spaces of smooth manifolds relative to a submersion; cf. Equation 0.6
(2)=differential algebraic jet spaces; cf. Equation 0.20
(3)=arithmetic jet spaces; cf. Equation 0.4

Here are a few more remarks on differential algebraic jet spaces. As in the case of smooth manifolds, the fibers of the maps $J^r(X) \to J^{r-1}(X)$ are affine spaces. Also there is a natural map at the level of $K-$points $\nabla : X(K) \to J^r(X)(K)$ naturally induced by the map $\nabla : \mathbf{A}^1(K) = K \to J^r(\mathbf{A}^1)(K) = K^{r+1}$ defined by $\nabla(a) = (a, \delta a, ..., \delta^r a)$. One defines the ring $\mathcal{O}^r(X)$ of $\delta-$*polynomial maps* on X to be the ring of all functions $f : X(K) \to K$ that can be written as $f = \tilde{f} \circ \nabla$ where $\tilde{f} \in \mathcal{O}(J^r(X))$ is a regular function on $J^r(X)$. One gets an isomorphism $\mathcal{O}^r(X) \simeq \mathcal{O}(J^r(X))$ of which the isomorphism in Equation 0.2 is the arithmetic analogue.

Next we address the following question: are the results of this book arithmetic analogues of results involving (genuine) differential equations ? In a certain loose sense this is indeed the case as explained below.

1) The spherical case of our theory here can be loosely viewed as an arithmetic analogue of some of the classical theory of differential invariants as found, for instance, in [**135**], [**114**].

2) The flat case of our theory here can be loosely viewed as an arithmetic analogue of the theory of the Manin map [**97**] and of the differential algebraic theory we developed in [**12**], [**14**], [**15**].

3) The hyperbolic case of our theory here can be loosely viewed as an arithmetic analogue of the classical theory of differential relations among modular forms and of the differential algebraic theory we developed in [**16**]; cf. also [**8**].

Let us provide some details.

0.3.2.1. *Spherical case.* We explain here a simple situation in the classical theory of differential invariants. Start with a group G acting by algebraic automorphisms on a variety X over K (which is usually affine). Then there is an induced action of G on the jet spaces $J^r(X)$. The ring of $G-$invariant functions $\mathcal{O}^r(X)^G = \mathcal{O}(J^r(X))^G$ corresponds to a special case of *differential invariants* in classical theory. A trivial example of this would be the natural action of $G := SL_2(C)$ on the affine plane $\mathbf{A}^2 = Spec\ K[x, y]$ over K; the Wronskian $xy' - yx' \in K[x, y, x', y'] = \mathcal{O}(J^1(\mathbf{A}^1))$ is then $G-$invariant. The spherical case of our arithmetic theory involves arithmetic analogues of computations of differential invariants in the above sense.

More generally the classical differential invariant theory deals with groups G (or simply vector fields V) acting directly on jet spaces $J^r(X)$ (in such a way that the action does not necessarily come "by functoriality" from an action on X itself). The situation is described in detail in Olver's book [**114**] in the setting of jet spaces in differential geometry.

0.3.2.2. *Flat case.* As already mentioned our flat theory will involve an arithmetic analogue of the theory of Manin maps [**97**], [**12**], [**14**], [**15**]. Let us quickly review the basics of the latter in the context of differential algebra following [**12**]. Assume X is an Abelian variety over K of dimension g. Then, by functoriality, each $J^r(X)$ has a natural structure of an algebraic group and is an extension of X by a vector group (i.e. a sum of copies of the additive group $\mathbf{G}_a = Spec\ K[t]$). By the theory of extensions of Abelian varieties by vector groups there is a surjective homomorphism $J^r(X) \to H$ onto a vector group H of dimension $\geq (r-1)g$. Consequently if $r \geq 2$ we have $H \neq 0$ hence $Hom(J^r(X), \mathbf{G}_a) \neq 0$. Elements of the latter composed with ∇ give rise to homomorphisms $\psi : X(K) \to K$. These homomorphisms are called *Manin maps*. They were discovered by Manin [**97**] (via a construction different from the one above) and were used by him to provide the first proof of the Mordell conjecture over function fields. Now the strategy presented above works well in the arithmetic case. The $p-$jet spaces $J^r(X_\wp)$ of an Abelian scheme $X_\wp$ over $R_\wp$ continue to be group objects and, although the kernel of the projection $J^r(X_\wp) \to \hat{X}_\wp$ ceases to be a sum of copies of $\hat{\mathbf{G}}_a := Spf\ R_\wp[t]\hat{}$, one can still prove that $Hom(J^r(X_\wp), \hat{\mathbf{G}}_a) \neq 0$ provided $r \geq 2$. As before elements of the latter yield $\delta-$functions $\psi : X(R_\wp) \to R_\wp$ which are homomorphisms; these should be viewed as arithmetic analogues of Manin's maps and play a key role in our theory in the flat case. Indeed maps of the form $\frac{\phi \circ \psi}{\psi}$ are invariant under the multiplication by integers N on our Abelian variety, $[N] : X_\wp \to X_\wp$; this invariance is the starting point for our use of ψ in our search for invariants in the flat case.

0.3.2.3. *Hyperbolic case.* We end our discussion of differential algebra by reviewing "differential algebraic invariants of isogenies" following [**16**]; this theory has an analogue in the arithmetic case that plays a key role in our treatment of the hyperbolic case. Let us consider the affine line $\mathbf{A}^1 = Spec\ K[j]$ viewed as a moduli space for elliptic curves over K. For any elliptic curve E over K we let $j(E) \in K$ be its $j-$invariant. On $K = \mathbf{A}^1(K)$ we have then an equivalence relation called *isogeny*: two points $a_1, a_2 \in K$ are called *isogenous* if there exists an isogeny of elliptic curves $E_1 \to E_2$ over K such that $j(E_1) = a_1$ and $j(E_2) = a_2$. The constant field $C \subset K$ is saturated with respect to isogeny hence so is $K \backslash C$. Then one can show [**16**] that there exists a rational function $U(j) \in \mathbf{Q}(j)$ such that if

$$\chi(j, j', j'', j''') := \frac{2j'j''' - 3(j'')^2}{4(j')^2} + (j')^2 U(j) \in K(j)[j', (j')^{-1}, j'', j'''] \tag{0.24}$$

then the function

$$K \backslash C \to K, \quad a \mapsto \chi(a, \delta a, \delta^2 a, \delta^3 a)$$

is constant on isogeny classes. (The fraction in Equation 0.24 is, of course, the classical Schwarzian operator.) We refer to [**16**] for more on this and for a higher dimensional generalization of this. It turns out that the theory behind the function χ above has an arithmetic analogue which leads to what we call $\delta-$*modular forms*; cf. [**21**], [**23**], [**24**]. These will be used heavily in this book.

0.3.3. Difference algebra. In the Ritt-Cohn difference algebra [**35**] one starts with a field K equipped with a ring endomorphism $\phi : K \to K$. For our discussion here we will allow K to have arbitrary characteristic. For any variety X over K one can then define a projective system of non-singular varieties

$$... \to J^r_\phi(X) \to J^{r-1}_\phi(X) \to ... \to J^0_\phi(X) = X$$

which can be called the *difference jet spaces* of X. If X is affine given by

$$X = Spec\ K[T]/(F)$$

where T is a tuple of indeterminates and F is a tuple of elements in $K[T]$ then $J^r_\phi(X)$ is, by definition, the spectrum

$$J^r_\phi(X) := Spec\ K[T, T^\phi, T^{\phi^2}, ..., T^{\phi^r}]/(F, \phi F, \phi^2 F, ..., \phi^r F) \tag{0.25}$$

where $T^\phi, T^{\phi^2}, ...$ are new tuples of variables and

$$\phi : K[T, T^\phi, ..., T^{\phi^{r-1}}] \to K[T, T^\phi, ..., T^{\phi^r}]$$

is the unique ring endomorphism prolonging $\phi : K \to K$ and sending $\phi T = T^\phi$, $\phi T^\phi = T^{\phi^2}$,... e.t.c. A basic difference between difference algebra and differential algebra is that the fibers of $J^r_\phi(X) \to J^{r-1}_\phi(X)$ are *not* affine spaces in general. Actually it is plain that $J^r_\phi(X)$ splits as a product of varieties

$$J^r_\phi(X) \simeq X \times X^\phi \times ... \times X^{\phi^r},$$

where X^{ϕ^i} is obtained from X by twisting the coefficients of the defining equations by ϕ^i. In particular, if X is definable over the fixed field of ϕ then $J^r_\phi(X)$ is just the $(r+1)-$fold product of X with itself. In spite of this geometric difference between difference algebra and differential algebra a lot of analogies between the two persist. Systems of linear equations, for instance, behave similarly [**62**], p. 20. At the non-linear level the geometric model theory of the two is strikingly similar; cf. especially the work of Hrushovski and Chatzidakis explained in [**69**].

It is interesting to compare difference algebra and arithmetic differential algebra. We refer to Remark **??** below for an argument suggesting that arithmetic differential algebra could be viewed as obtained from difference algebra by adjoining certain divergent series. Here we close our discussion by indicating a possible bridge at the level of jet spaces. Let us assume, for simplicity, that X is an affine scheme given by the spectrum of $R_\wp[T]/(F)$ as in Equation 0.3. Let $K_\wp$ be the field of fractions of $R_\wp$. Then one can consider a canonical algebraisation of the $p-$jet space $J^r(X_\wp)$ given as the scheme

$$\mathcal{J}^r(X_\wp) := Spec\ R_\wp[T, T', T'', ..., T^{(r)}]/(F, \delta F, \delta^2 F, ..., \delta^r F). \tag{0.26}$$

Then clearly the $p-$adic completion of $\mathcal{J}^r(X_\wp)$ is $J^r(X_\wp)$; in symbols:

$$\mathcal{J}^r(X_\wp)\hat{} = J^r(X_\wp). \tag{0.27}$$

On the other hand it is trivial to see that the base change to $K_\wp$ of $\mathcal{J}^r(X_\wp)$ is $J^r_\phi(X_\wp \otimes K_\wp)$; in symbols,

$$\mathcal{J}^r(X_\wp) \otimes K_\wp = J^r_\phi(X_\wp \otimes K_\wp). \tag{0.28}$$

By Equations 0.27 and 0.28 we see that $\mathcal{J}^r(X_\wp)$ might be viewed as a "bridge" between the jet spaces of difference algebra and arithmetic differential algebra.

Whether this bridge can actually be crossed is an open question. We can summarize the above discussion by adjoining to the diagram in Equation 0.23 the diagram:

$$\begin{array}{ccc} (2) & --- & (3) \\ | & & | \\ | & & | \\ (4) & --- & (5) \end{array} \tag{0.29}$$

(2)=differential algebraic jet spaces; cf. Equation 0.20
(3)=arithmetic jet spaces; cf. Equation 0.4
(4)=difference jet spaces; cf. Equation 0.25
(5)=canonical algebraisation of arithmetic jet spaces; cf. Equation 0.26.

0.3.4. Dynamical systems. Let us start by reviewing some aspects of the classical iteration theory initiated by Fatou and Julia [**109**]. In this theory one starts with a rational map $s : \mathbf{P}^1 \to \mathbf{P}^1$ from the complex projective line into itself and one studies the behavior of orbits

$$z, s(z), s^2(z), ...$$

as z moves in $\mathbf{P}^1$. The z's for which the orbit is "unstable" form a closed set called the *Julia set*. The complement of the Julia set is called the *Fatou set*. Points that eventually return to themselves after a number of iterations are called *periodic*. Up to finitely many points the Julia set turns out to coincide with the closure of the set of periodic points and usually looks like a "fractal". A point $z \in \mathbf{C}$ is critical if the tangent map of s at z vanishes. The orbits of critical points hold the key to many of the dynamical properties of s. In particular the maps s for which all critical points have finite orbits have been closely investigated by Thurston [**51**]; they are called *postcritically finite* and one can attach orbifolds and Euler characteristics to them. Postcritically finite maps with Euler characteristic 0 have been classified and they are essentially those that admit an "analytic uniformization" by the complex plane $\mathbf{C}$. Let us call them here, for simplicity, *flat maps*. Flat maps can be explicitly described as being either multiplicative maps, or Chebyshev polynomials or *Lattès maps* (the latter being maps induced by endomorphisms of elliptic curves). The Julia sets of flat maps are "smooth" (i.e. not "fractals"). For multiplicative maps the Julia set is a circle. For Chebyshev maps the Julia set is a segment. For Lattès maps the Julia set is the whole of $\mathbf{P}^1$. (Note however that flat maps are not the only maps that have smooth Julia sets !)

Flat maps play a key role in our theory. Indeed let us attach to any rational map s a correspondence $\mathbf{X}^* = (\mathbf{P}^1, \mathbf{P}^1, id, s)$ in complex algebraic geometry. Assume, for simplicity, that s has rational coefficients. Fix a prime p. Then, as explained earlier, we can (roughly speaking) attach to $\mathbf{X}$ a correspondence $\mathbf{X}_\delta$ in $\delta-$geometry at p. What we will prove in this book is, roughly speaking, that the categorical quotient of $\mathbf{X}_\delta$ in $\delta-$geometry is non-trivial for almost all p if and only if s is a flat map.

More generally the concept of postcritical finiteness (and a suitable generalization of it to the case of correspondences that do not come from dynamical systems) plays a key role in the present book. This will be explained in detail in Chapters 1 and 2.

We close our discussion here by noting that complex dynamics has an interesting $p-$adic analogue. For instance, the Lubin-Tate theory of formal groups [**92**], [**93**] can be viewed as an early incarnation of a non-archimedian dynamics. For more recent work see, for instance [**93**], [**5**], [**6**], [**91**], and the bibliographies therein. Although our theory lives in the $p-$adic world and although we *will* use formal groups at periodic points in our theory, we will not need to use, in our book, the "genuine" theory of $p-$adic dynamical systems in the above cited papers. Here, by "genuine" we mean "dealing with the $p-$adic Julia set, $p-$adic wandering Fatou domains, etc." (On the other hand, as already mentioned, we *do* need to use the theory of *complex* dynamical systems through the orbifold characterization of flat maps.) Needless to say it would be very interesting to make our theory interact with the "genuine" theory of $p-$adic dynamical systems.

0.3.5. Connes' non-commutative geometry. The approach in the present book seems to be, in some sense, perpendicular to Connes' non-commutative geometric approach to quotient spaces; cf. our previous remarks about invariant versus groupoid ideology. Nevertheless, since some examples occur in both theories it is not unreasonable to expect interactions between the two approaches. In what follows we shall discuss some general principles of non-commutative geometry and we shall examine some examples. Our discussion will inevitably be extremely superficial and not entirely precise; for precise, in-depth presentations we refer to [**36**], [**99**], [**101**].

Connes' theory is formulated in the context of functional analysis and, although some parts of it have a more algebro-geometric flavor [**127**] [**99**], our presentation will follow Connes' original approach. As already mentioned the starting point in this theory is often a groupoid

$$\mathcal{G} = (X, \tilde{X}, \sigma_1, \sigma_2, \nu, \iota, \epsilon)$$

in some "geometric" category, for instance the category of smooth manifolds. Write $\gamma_1\gamma_2 = \nu(\gamma_1, \gamma_2)$. The first step is to attach to $\mathcal{G}$ a C^*-algebra, $C^*(\mathcal{G})$, called the *convolution algebra* of $\mathcal{G}$. Naively $C^*(\mathcal{G})$ might be thought of as (a norm completion of) some algebra of complex valued functions f on $\tilde{X}$ with multiplication given by "convolution":

$$(f_1 \star f_2)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f_1(\gamma_1) f_2(\gamma_2).$$

As it is this cannot work because the sums involved are usually infinite. So one replaces, according to the context, sums by integrals, functions by "densities", etc.

The next step in Connes' theory is the realization that "non-commutative spaces" should not be defined as C^*-algebras but rather as *Morita equivalence classes* of C^*-algebras. Two C^*-algebras A and B are *Morita equivalent* if there exists a bimodule ${}_AM_B$ with certain extra data that allow one to "transfer" modules between A and B. So, roughly speaking, one defines the *non-commutative space* attached to the groupoid $\mathcal{G}$ as the C^*-algebra $C^*(\mathcal{G})$ up to Morita equivalence.

At this point one is faced with the "topological and geometrical study of non-commutative spaces"; this constitutes the heart of the theory and, for this purpose, a whole spectrum of techniques is brought into the picture such as: spectral theory, index theory, $K-$theory, etc.

In what follows we examine a number of examples in non-commutative geometry and compare them with the spherical, flat, and hyperbolic examples we are able to treat in our theory.

0.3.5.1. *Spherical case.* Consider the action of $PSL_2(\mathbf{Z})$ on $\mathbf{P}^1(\mathbf{C})$ by homographies. Note that

$$\mathbf{P}^1(\mathbf{C}) = \mathbf{H}^{\pm} \cup \mathbf{P}^1(\mathbf{R}), \quad \mathbf{H}^{\pm} := \mathbf{C} \backslash \mathbf{R} = \mathbf{H}^+ \cup \mathbf{H}^-,$$

$\mathbf{H}^+ = \mathbf{H}$ is the upper half plane in $\mathbf{C}$, $\mathbf{H}^-$ is the lower half plane, and $\mathbf{P}^1(\mathbf{R})$ is the real projective line. The action of $PSL_2(\mathbf{Z})$ on $\mathbf{H}^{\pm}$ is properly discontinuous and the quotient exists in the category of Riemann surfaces; it is the union of 2 copies of the modular curve $\mathbf{H}/PSL_2(\mathbf{Z})$ parameterizing elliptic curves over the complex numbers. However the action of $PSL_2(\mathbf{Z})$ on the "stratum" $\mathbf{P}^1(\mathbf{R})$ is not properly discontinuous and actually the categorical quotient in the category of manifolds reduces to a point. What can be done in non-commutative geometry is to consider the groupoid $\mathcal{G}$ attached to the action

$$PSL_2(\mathbf{Z}) \times \mathbf{P}^1(\mathbf{R}) \to \mathbf{P}^1(\mathbf{R})$$

and consider the C^*-algebra $C^*(\mathcal{G})$ modulo Morita equivalence. The latter is declared to *be*, by definition, the "quotient" $\mathbf{P}^1(\mathbf{R})/PSL_2(\mathbf{Z})$ and can be interpreted, in a natural way as a compactification of the modular curve $\mathbf{H}/PSL_2(\mathbf{Z})$; cf. the remarks in the next example. The non-commutative space $\mathbf{P}^1(\mathbf{R})/PSL_2(\mathbf{Z})$ has been studied by Manin and Marcolli [**100**] who proved a number of deep results about "limiting" behavior of objects from the usual modular curve $\mathbf{H}/PSL_2(\mathbf{Z})$ that tend to points on the boundary.

On the other hand the space of orbits of $\mathbf{P}^1$ under the action of $PSL_2(\mathbf{Z})$ will be one of the main examples of our theory in the spherical case; cf. the present Introduction. It would be beautiful to understand if there is a connection between the two approaches. Remark that from our perspective $\mathbf{P}^1$ modulo $PSL_2(\mathbf{Z})$ is in a "spherical situation" while in non-commutative geometry $\mathbf{P}^1(\mathbf{R})$ modulo $PSL_2(\mathbf{Z})$ is at the boundary of a "hyperbolic commutative geometric object".

0.3.5.2. *Flat case.* One of the first remarkable objects studied in non-commutative geometry were the non-commutative tori of Connes and Rieffel [**36**], [**40**]. There are analogies between these objects and the objects relevant in the flat case of our theory. Let us briefly explain this in the case of "complex dimension" one. A complex torus of dimension one is an elliptic curve E over $\mathbf{C}$. One can represent E as the quotient E_τ (in the category of Riemann surfaces) of the complex plane $\mathbf{C}$ by the action (by translation) of the subgroup $\mathbf{Z}+\tau\mathbf{Z}$, where $\tau \in \mathbf{H}$. Alternatively one can describe E_τ as the quotient (in the category of Riemann surfaces) of $\mathbf{C}^\times$ by the action of the subgroup $\langle q_\tau \rangle$ where $q_\tau = e^{2\pi i \tau}$. As τ approaches a point $\theta \in \mathbf{R} \backslash \mathbf{Q}$, q_τ will approach $q_\theta := e^{2\pi i \theta}$ which is on the unit circle $S^1 \subset \mathbf{C}$, but not a root of unity. In non-commutative geometry it is possible to define an object which can be viewed as the limit of a family of elliptic curves E_τ when $\tau \to \theta$. Here is the construction. One considers the action

$$\mu_\theta : \mathbf{Z} \times S^1 \to S^1, \quad \mu_\theta(m, z) := q_\theta^m \cdot z,$$

one attaches to this action a groupoid $\mathcal{G}_\theta$ and one defines the C^*-algebra $A_\theta := C^*(\mathcal{G}_\theta)$; the Morita equivalence class of A_θ is interpreted as a *non-commutative elliptic curve* which is the *limit of* E_τ *as* $\tau \to \theta$. One of the crucial early discoveries in non-commutative geometry was that for two irrational real numbers θ_1 and θ_2 the C^*-algebras A_{θ_1} and A_{θ_2} are Morita equivalent if and only if θ_1 and θ_2 are $PSL_2(\mathbf{Z})-$conjugate. This justifies the claim in the previous example that

$\mathbf{P}^1(\mathbf{R})/PSL_2(\mathbf{Z})$ can be interpreted as a compactification of the modular curve $\mathbf{H}/PSL_2(\mathbf{Z})$. (Indeed $\mathbf{P}^1(\mathbf{Q})/PSL_2(\mathbf{Z})$ consists of one point, the classical *cusp*.)

Now let us note an analogy between the above example and what will be involved in the flat case of our theory. The quotient of S^1 by the action ρ_θ is a quotient $S^1/\langle q_\theta\rangle$ of the Lie group S^1 by a cyclic subgroup (which acts "wildly"). Accordingly one can propose to study (categorical) quotients of the form $G/\langle g_1,...,g_n\rangle$ where G is a commutative algebraic group and $\langle g_1,...,g_n\rangle$ is a finitely generated subgroup. Such quotients have actually been studied in differential algebra [**12**], [**13**] in relation to the Mordell conjecture over function fields and their study involves the Manin maps discussed before. As already mentioned Manin maps have arithmetic analogues that play a key role in this book.

0.3.5.3. *Hyperbolic case.* Hecke correspondences play an important role in noncommutative geometry; cf. [**39**], [**37**], [**38**], [**101**]. On the other hand Hecke correspondences will be at the heart of the hyperbolic case of the present book. We would like to compare the two theories in what follows. Let, as usual, $\mathbf{A}_f$ denote the finite adeles of $\mathbf{Q}$ i.e.

$$\mathbf{A}_f := \left(\lim_{\leftarrow} \frac{\mathbf{Z}}{n\mathbf{Z}}\right) \otimes \mathbf{Q}.$$

Then, following Shimura, one considers the space

$$Sh = GL_2(\mathbf{Q})\backslash(GL_2(\mathbf{A}_f) \times \mathbf{H}^{\pm}). \tag{0.30}$$

For any congruence subgroup $\Gamma \subset SL(2,\mathbf{Z})$ consider the modular curve

$$Sh_\Gamma := \mathbf{H}/\Gamma,$$

where we wrote $\mathbf{H}/\Gamma$ instead of $\Gamma\backslash\mathbf{H}$ in order to make notation agree with notation in the present book. One can consider then the inverse limit

$$Sh^0 := \lim_{\leftarrow} Sh_\Gamma$$

over all congruence subgroups $\Gamma \subset SL_2(\mathbf{Z})$. The space Sh^0 turns out to be a connected component of Sh. All this is classical, i.e. commutative. Now, following [**38**] one can embed the set Sh into the "much larger" set

$$Sh^{(nc)} := GL_2(\mathbf{Q})\backslash(M_2(\mathbf{A}_f) \times \mathbf{H}^{\pm}).$$

The set $Sh^{(nc)}$ does not have a nice "commutative space" structure but one can attach to it a C^*-algebra $\mathcal{A}_2$ starting from a certain (alternative !) presentation of $Sh^{(nc)}$ as a quotient by an action; cf. [**101**], p. 67. We refer to loc. cit. for an overview of the beautiful and deep theory of $\mathcal{A}_2$ and also for an overview of the Bost-Connes $\mathcal{A}_1$ that was its GL_1 prototype.

Now turning to our theory let us consider a prime number l, the matrix

$$\tau_l := \left[\begin{array}{cc} l & 0 \\ 0 & 1 \end{array}\right],$$

and the subgroup

$$\langle \Gamma, \tau_l\rangle \subset GL_2(\mathbf{Q})$$

(where $\Gamma \subset SL_2(\mathbf{Z})$ is any congruence subgroup). Define the set

$$Sh_{\langle\Gamma,\tau_l\rangle} := \mathbf{H}/\langle\Gamma,\tau_l\rangle.$$

This set should be viewed, roughly, as the quotient of Sh_Γ by a Hecke correspondence. Since the group $\langle\Gamma,\tau_l\rangle$ does not act properly discontinuously on $\mathbf{H}$ the set

$Sh_{\langle\Gamma,\tau_l\rangle}$ is not an object of analytic geometry anymore. Our theory can be viewed as a way to put on the set $Sh_{\langle\Gamma,\tau_l\rangle}$ a geometric structure in $\delta-$geometry. Summarizing we have the following inclusions of sets

$$Sh^0 \subset Sh \subset Sh^{(nc)}$$

and the following projections of sets

$$Sh^0 \to Sh_\Gamma \to Sh_{\langle\Gamma,\tau_l\rangle}.$$

The set $Sh^{(nc)}$ is replaced, in non-commutative geometry, by the C^*-algebra $\mathcal{A}_2$ while the set $Sh_{\langle\Gamma,\tau_l\rangle}$ is replaced, in our theory, by an object of $\delta-$geometry. It is plain from the above discussion that the non-commutative theory and our theory "diverge" in this example. The set we are interested in, $Sh_{\langle\Gamma,\tau_l\rangle}$, is a "wild" quotient of the "commutative part" of $Sh^{(nc)}$.

0.3.6. Drinfeld modules. The theory of Drinfeld modules [**52**], [**62**], [**130**] exhibits analogies with our theory that deserve being understood. Let us quickly recall the definition of a Drinfeld module and make a few comments. One starts with a smooth projective geometrically connected curve X over a finite field $\mathbf{F}_{p^m}$ equipped with a point $\infty \in X$. Let $\mathbf{A}$ be the ring of regular functions on $X\backslash\{\infty\}$. Let $\iota : \mathbf{A} \to A$ be a homomorphism into a field A. Let $F : A \to A$ be the p^m-power homomorphism. Let $A[F]$ be the non-commutative ring generated by A and a variable, called F, with relations $F \cdot a = F(a) \cdot F = a^{p^m} \cdot F$, $a \in A$. (Morally $A[F]$ is a *convolution algebra* attached to the action of F on A.) Let $e : A[F] \to A$ be the ring homomorphism sending $e(\sum a_i F^i) = a_0$ and let $j : A \to A[F]$ be the inclusion. Then a *Drinfeld module* is an $\mathbf{F}_{p^m}-$algebra homomorphism $\psi : \mathbf{A} \to A[F]$ such that $e \circ \psi = \iota$, $\psi \neq j \circ \iota$. The ring $A[F]$ acts on A (with the letter F acting on A as the endomorphism F and A acting on A by multiplication). Hence $\mathbf{A}$ will act on A via ψ. The theory of Drinfeld modules can be viewed therefore as a very non-trivial example of "algebra / geometry with operators"; it entirely lives in characteristic p. The ring $\mathbf{A}$ is viewed, in this theory, as an analogue of the ring $\mathbf{Z}$ of integers. Its quotient field $\mathbf{k}$ plays the role of $\mathbf{Q}$. The completion $\mathbf{K}$ of $\mathbf{k}$ at ∞ plays the role of the real field, $\mathbf{R}$. The completion $\mathbf{C}_\infty$ of an algebraic closure of $\mathbf{K}$ plays the role of the complex field $\mathbf{C}$. The "first" examples of the theory are Drinfeld modules $\psi : \mathbf{A} \to \mathbf{C}_\infty[F]$ attached to "lattices". Drinfeld modules of ranks 1 and 2 play the role of one dimensional algebraic groups (multiplicative group and elliptic curves respectively), viewed as $\mathbf{Z}-$modules (or sometimes modules over rings of integers in imaginary quadratic fields). They can also be viewed as analogues of formal groups viewed as $\mathbf{Z}-$modules (or sometimes modules over rings of integers in $p-$adic fields). There are analogies between the exponentials in the theory of Drinfeld modules and our arithmetic Manin maps; both classes of maps have, as kernels, objects that can be viewed as "lattices". It is a suggestion of Manin (cf. private communication to the author) that maybe there exists a lifting to characteristic zero of Drinfeld module theory which is similar in spirit to the arithmetic differential algebra of the present book.

0.3.7. Dwork's theory. It is interesting to compare our *theory of arithmetic differential equations* with what is usually understood by the *arithmetic theory of differential equations* (as it appears in the work of Dwork, for instance; cf. [**55**]). It turns out that the two theories are "perpendicular" in the very precise sense that they deal with "differentiation" in two "perpendicular" directions. Indeed

Dwork's theory is about *genuine* differential equations (i.e. equations involving d/dt) whereas our theory is about *analogues* of differential equations (involving, instead, the Fermat quotient operator δ). In a precise sense d/dt and δ point in two different directions. Now although the two theories are perpendicular this doesn't mean they don't interact. Indeed part of Dwork's theory can be rephrased in terms of crystalline cohomology; cf. the report by Katz [**82**]. The genuine differential equations can be read off the Gauss-Manin connection whereas our arithmetic differential equations are closely related to the action of Frobenius. On the other hand Gauss-Manin and Frobenius are, as well known, both present in the crystalline picture. We will actually heavily rely in our proofs on this crystalline interpretation when we get to discuss the hyperbolic case.

0.3.8. Mochizuchi's $p-$adic Teichmüller theory. An interesting problem is to find interactions between our theory here and Mochizuchi's $p-$adic Teichmüller theory [**111**]. Unlike the Tate and Mumford $p-$adic uniformizations of curves with (totally) degenerate reduction mod p both our theory and Mochizuchi's can be viewed, in a certain sense, as "uniformization" theories in the case of good reduction. As an example of what this might mean in the case of our theory we propose to see the "inverse" of our arithmetic Manin map of an elliptic curve E (cf. our discussion above) as a multivalued map from the affine line $\mathbf{A}^1$ into E. This can be interpreted, in some bold sense, as a sort of uniformization of E.

0.3.9. Ihara's congruence relations. There an interesting possibility that a profitable link can be established between our approach here and Ihara's beautiful work on congruence relations (i.e. liftings to characteristic zero of the correspondence $\Pi \cup \Pi'$ on a curve of characteristic p, where Π is the graph of the Frobenius); cf. [**73**], [**74**]. A hint as to such a link can be seen, for instance, in Lemma **??** of this book. On the other hand we would like to point out what we think is an important difference between our viewpoint here and the viewpoint proposed by Ihara in a related paper [**75**]. Our approach, in its simplest form, proposes to see the operator

$$\delta = \delta_p : \mathbf{Z} \to \mathbf{Z},\ a \mapsto \delta a = \frac{a - a^p}{p},$$

where p is a fixed prime, as an analogue of a derivation with respect to p. In [**75**] Ihara proposed to see the map

$$d : \mathbf{Z} \to \prod_p \mathbf{Z}/p\mathbf{Z},\ a \mapsto \left(\frac{a - a^p}{p}\ mod\ p \right) \tag{0.31}$$

as an analogue of differentiation for integers and he proposed a series of very interesting conjectures concerning the "zeroes" of the differential of an integer; these conjectures are still completely open. The main difference between Ihara's viewpoint and ours is that *we do not consider the reduction mod p of the Fermat quotients but the Fermat quotients themselves.* This allows the possibility of considering iterates δ^r of our δ which leads to the possibility of considering arithmetic analogues of higher order differential equations; and indeed most such equations relevant to our theory will have order ≥ 2. On the other hand it doesn't make sense to consider iterates of Ihara's operator d. One way out of this dilemma would be to find a map $D : \mathbf{Z} \to \mathbf{Z}$ which, composed with the canonical projection $\mathbf{Z} \to \prod_p \mathbf{Z}/p\mathbf{Z}$, yields Ihara's map d. (Then one could consider iterates D^r of D.) Voloch [**132**] proved, however, that no such map D can exist at least if one assumes the conjecture that there are only finitely many Mersenne primes.

0.3.10. Field with one element. The theory proposed in this book seems to agree with certain aspects of the "myth of the field with one element" emerging from the work of Kurokawa, Soulé, Deninger, Manin, and others; cf. [**86**], [**128**], [**98**], [**46**]. For instance the constants $a \in \hat{\mathbf{Z}}_p^{ur}$, $\delta a = 0$, of the map $\delta : \hat{\mathbf{Z}}_p^{ur} \to \hat{\mathbf{Z}}_p^{ur}$ consist of the roots of unity in $\hat{\mathbf{Z}}_p^{ur}$ together with 0; on the other hand the roots of unity and 0 are, according to the philosophy of the field with one element, precisely the $\bar{\mathbf{F}}_1-$ points of the affine line where $\bar{\mathbf{F}}_1$ is the "algebraic closure of the field with one element, $\mathbf{F}_1$". Not all the predictions of the philosophy of the field with one element seem however to be in agreement with our theory. For instance there seems to be an important ideological difference between our operator

$$\delta_p : \mathbf{Z} \to \mathbf{Z}, \quad \delta_p a = \frac{a - a^p}{p},$$

and the *absolute derivation* introduced in [**86**],

$$\frac{\partial}{\partial p} : \mathbf{Z} \to \mathbf{Z}, \quad \frac{\partial}{\partial p}(p^e m) := e p^{e-1} m,$$

where $m \in \mathbf{Z}$ is prime to p. Indeed the constants of δ_p are 0 and ± 1 while the constants of $\partial/\partial p$ are 0 and all the integers coprime to p. This suggests that, according to the philosophy of $\mathbf{F}_1$, there is no theory of $\mathbf{F}_1$ that involves *one prime only*; the only theory about $\mathbf{F}_1$ would involve *all the primes at the same time.* On the contrary, in our book, there is a theory for each individual prime.

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