

Foundations of arithmetic differential geometry

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Preface

Aim of the book. The aim of this book is to introduce and develop an arithmetic analogue of *classical differential geometry*; this analogue will be referred to as *arithmetic differential geometry*. In this new geometry the ring of integers \mathbb{Z} will play the role of a ring of functions on an infinite dimensional manifold. The role of coordinate functions on this manifold will be played by the prime numbers $p \in \mathbb{Z}$. The role of partial derivatives of functions with respect to the coordinates will be played by the *Fermat quotients*, $\delta_p n := \frac{n-n^p}{p} \in \mathbb{Z}$, of integers $n \in \mathbb{Z}$ with respect to the primes p . The role of metrics (respectively 2-forms) will be played by symmetric (respectively anti-symmetric) matrices with coefficients in \mathbb{Z} . The role of connections (respectively curvature) attached to metrics or 2-forms will be played by certain adelic (respectively global) objects attached to matrices as above. One of the main conclusions of our theory will be that (the “manifold” corresponding to) \mathbb{Z} is “intrinsically curved;” this curvature of \mathbb{Z} (and higher versions of it) will be encoded into a \mathbb{Q} -Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$, which we refer to as the *holonomy algebra*, and the study of this algebra is, essentially, the main task of the theory.

Needless to say, *arithmetic differential geometry* is still in its infancy. However, its foundations, which we present here, seem to form a solid platform upon which one could further build. Indeed, the main differential geometric concepts of this theory turn out to be related to classical number theoretic concepts (e.g., Christoffel symbols are related to Legendre symbols); existence and uniqueness results for the main objects (such as the arithmetic analogues of Ehresmann, Chern, Levi-Civita, and Lax connections) are being proved; the problem of defining curvature (which in arithmetic turns out to be non-trivial) is solved in some important cases (via our method of analytic continuation between primes and, alternatively, via algebraization by correspondences); and some basic vanishing/non-vanishing theorems are being proved for various types of curvature. It is hoped that all of the above will create a momentum for further investigation and further discovery.

Immediate context. A starting point for this circle of ideas can be found in our paper [23] where we showed how to construct arithmetic analogues of the classical jet spaces of Lie and Cartan; these new spaces were referred to as *arithmetic jet spaces*. The main idea, in this construction, was to replace classical derivation operators acting on functions with *Fermat quotient operators* acting on numbers and to develop an *arithmetic differential calculus* that parallels classical calculus. There were two directions of further development: one towards a theory of *arithmetic differential equations* and another one towards an *arithmetic differential geometry*. A theory of *arithmetic differential equations* was developed in a series of papers [23]-[42], [7] and was partly summarized in our monograph [35] (cf. also the survey papers [43, 102]); this theory led to a series of applications to invariant

theory [27, 7, 28, 35], congruences between modular forms [27, 37, 38], and Diophantine geometry of Abelian and Shimura varieties [24, 36]. On the other hand an *arithmetic differential geometry* was developed in a series of papers [42]-[47], [8]; the present book follows, and further develops, the theory in this latter series of papers.

We should note that our book [35] on *arithmetic differential equations* and the present book on *arithmetic differential geometry*, although both based on the same conceptual framework introduced in [23], are concerned with rather different objects. In particular the two books are independent of each other and the overlap between them is minimal. Indeed the book [35] was mainly concerned with arithmetic differential calculus on Abelian and Shimura varieties. By contrast, the present book is concerned with arithmetic differential calculus on the classical groups

$$GL_n, SL_n, SO_n, Sp_n,$$

and their corresponding symmetric spaces.

Of course, the world of Abelian/Shimura varieties and the world of classical groups, although not directly related within *abstract algebraic geometry*, are closely related through *analytic* concepts such as *uniformization* and *representation theory*. The prototypical example of this relation is the identification

$$M_1(\mathbb{C}) \simeq SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO_2(\mathbb{R})$$

where M_1 is the coarse moduli space of Abelian varieties

$$\mathbb{C}/\text{lattice}$$

of dimension one; the curve M_1 is, of course, one of the simplest Shimura varieties. It is conceivable that the analytic relation between the two worlds referred to above could be implemented via certain *arithmetic differential correspondences* (correspondences between arithmetic jet spaces); in particular it is conceivable that the subject matter of the present book and that of our previous book [35] might be related in ways that go beyond what can be seen at this point. A suggestion for such a possible relation comes from the theory of δ -Hodge structures developed in [22] and from a possible arithmetic analogue of the latter that could emerge from [7, 35].

There are at least two major differential geometric themes that are missing from the present book: *geodesics* and the *Laplacian*. It is unclear at this point what the arithmetic analogues of geodesics could be. On the other hand, for first steps towards an arithmetic Laplacian (and, in particular, for a concept of curvature based on it), we refer to [34]. But note that the flavor of [34] is quite different from that of the present book. Again, it is conceivable that the arithmetic Laplacian theory in [34] could be related, in ways not visible at this point, with our theory here.

Larger context. By what has been said so far this book is devoted to unveiling a new type of “geometric” structures on \mathbb{Z} . This is, of course, in line with the classical effort of using the analogy between numbers and functions to the advantage of number theory; this effort played a key role in the development of number theory throughout the 20th century up to the present day, as reflected in the work of Dedekind, Kronecker, Hilbert, Artin, Weil, Iwasawa, Grothendieck,

Manin, Parshin, Arakelov (to name just a few of the early contributors). However, as we shall explain in the Introduction, this classical theory (which views number fields as analogous to Riemann surfaces) differs in essential ways from the theory of the present book (which views number fields as analogous to infinite dimensional manifolds). Our theory should also be contrasted with more recently proposed paradigms for “the discrete” such as: arithmetic topology [104]; the theory around the p -curvature conjecture of Grothendieck [87]; the Ihara differential [78]; the Fontaine-Colmez calculus [64]; discrete differential geometry [10]; and the geometries over the field with one element, \mathbb{F}_1 , proposed by a number of people including: Tits [118], Kapranov-Smirnov [86], Soulé [115], Deitmar [56], Kurokawa et.al. [93], Connes-Consani [55], Manin-Marcolli [117], Lorscheid [117], Haran [70], etc. The non-vanishing curvature in our theory also prevents our arithmetic differential geometry from directly fitting into Borger’s λ -ring framework [13] for \mathbb{F}_1 ; indeed, roughly speaking, λ -ring structure leads to zero curvature. For each individual prime, though, our theory is consistent with Borger’s philosophy of \mathbb{F}_1 ; cf. [13, 102] and the Introduction to [35]. In spite of the differences between these theories and ours we expect interesting analogies and interactions.

This being said what is, after all, the position of our theory among more established mathematical theories? The answer we would like to suggest is that our curvature of \mathbb{Z} , encoded in the Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$, could be viewed as an infinitesimal counterpart of the absolute Galois group $\Gamma_{\mathbb{Q}} := \text{Gal}(\mathbb{Q}^a/\mathbb{Q})$ of \mathbb{Q} : our Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$ should be to the absolute Galois group $\Gamma_{\mathbb{Q}}$ what the identity component Hol^0 of the holonomy group Hol is to the monodromy group Hol/Hol^0 in classical differential geometry. As such our $\mathfrak{hol}_{\mathbb{Q}}$ could be viewed as an object of study in its own right, as fundamental, perhaps, as the absolute Galois group $\Gamma_{\mathbb{Q}}$ itself. A “unification” of $\mathfrak{hol}_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ may be expected in the same way in which Hol^0 and Hol/Hol^0 are “unified” by Hol . Such a unification of $\mathfrak{hol}_{\mathbb{Q}}$ with $\Gamma_{\mathbb{Q}}$ might involve interesting Galois representations in the same way in which the unification of $\Gamma_{\mathbb{Q}}$ with the geometric fundamental group via the arithmetic fundamental group in Grothendieck’s theory yields interesting Galois representations [77, 57].

Organization of the book. The book starts with an Introduction in which we give an outline of our theory and we briefly compare our theory with other theories.

In Chapter 1 we briefly review the basic algebra and superalgebra concepts that we are going to use throughout the book. Most of this material is standard so the reader may choose to skip this chapter and consult it later as needed.

Chapter 2 will be devoted to revisiting classical differential geometry from an algebraic standpoint. We will be using the language of differential algebra [91, 112], i.e., the language of rings with derivations. We call the attention upon the fact that some of the classical differential geometric concepts will be presented in a somewhat non-conventional way in order to facilitate and motivate the transition to the arithmetic setting. This chapter plays no role in our book other than that of a motivational and referential framework; so, again, the reader may choose to skip this chapter and then go back to it as needed or just in order to compare the arithmetic theory with (the algebraic version of) the classical one.

Chapter 3 is where the exposition of our theory properly begins; here we present the basic notions of arithmetic differential geometry and, in particular, we introduce

our arithmetic analogues of connection and curvature. The theory will be presented in the framework of arbitrary (group) schemes.

Chapter 4 specializes the theory in Chapter 3 to the case of the group scheme GL_n ; here we prove, in particular, our main existence results for Ehresmann, Chern, Levi-Civita, and Lax connections respectively.

Chapters 5, 6, 7, 8 are devoted to the in-depth analysis of these connections; in particular we prove here the existence of the analytic continuation between primes necessary to define curvature for these connections and we give our vanishing/non-vanishing results for these curvatures. In Chapter 5, we also take first steps towards a corresponding (arithmetic differential) Galois theory.

The last Chapter 9 lists some of the natural problems, both technical and conceptual, that one faces in the further development of the theory.

Chapters 1, 2, 3, 4 should be read in a sequence (with Chapters 1 and 2 possibly skipped and consulted later as needed); Chapters 5, 6, 7, 8 depend on Chapters 1, 2, 3, 4 but are essentially independent of each other. Chapter 9 can be read right after the Introduction.

Cross references are organized in a series of sequences as follows. Sections are numbered in one sequence and will be referred to as “section $x.y$.” Definitions, Theorems, Propositions, Lemmas, Remarks, and Examples are numbered in a separate sequence and are referred to as “Theorem $x.y$, Example $x.y$,” etc. Finally equations are numbered in yet another separate sequence and are referred to simply as “ $x.y$.” For all three sequences x denotes the number of the chapter.

Readership and prerequisites. The present book addresses graduate students and researchers interested in algebra, number theory, differential geometry, and the analogies between these fields. The only prerequisites are some familiarity with commutative algebra (cf., e.g., the Atiyah-MacDonald book [4] or, for more specialized material, Matsumura’s book [103]) and with foundational scheme theoretic algebraic geometry (e.g., the first two chapters of Hartshorne’s book [72]). The text also contains a series of remarks that assume familiarity with basic concepts of classical differential geometry (as presented in [89], for instance); but these remarks are not essential for the understanding of the book and can actually be skipped.

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Introduction

By *classical differential geometry* we will understand, in this book, the theory of differentiable manifolds equipped with various “geometric structures” such as metrics, connections, curvature, etc. Our standard reference for this theory is [89]. However we shall only be interested in the algebraic aspects of this theory so we will allow ourselves below to reintroduce some of its concepts using the language of differential algebra [91, 112] (i.e., the language of rings equipped with derivations) and we will still refer to this “algebraized” version of differential geometry as *classical differential geometry*. On the other hand our purpose, in this book, is to develop an arithmetic analogue of classical differential geometry, which we will refer to as *arithmetic differential geometry*; in this latter theory derivation operators acting on functions will be replaced by Fermat quotient type operators acting on numbers, in the spirit of [23, 35]. In this Introduction we present an outline of our theory and we provide a quick comparison of our theory with other theories.

0.1. Outline of the theory

0.1.1. Classical versus arithmetic differentiation. In the classical theory of smooth (i.e., differentiable) manifolds one considers, for any m -dimensional smooth manifold M , the ring of smooth real valued functions $C^\infty(M, \mathbb{R})$. For the purposes of this Introduction it is enough to think of M as being the Euclidean space $M = \mathbb{R}^m$. In this book the arithmetic analogue of the manifold \mathbb{R}^m will be the scheme $\text{Spec } \mathbb{Z}$. In order to stress our analogy between functions and numbers it is convenient to further fix a subring

$$(0.1) \quad A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$$

that is stable under partial differentiation and to frame the classical differential geometric definitions in terms of this ring. The arithmetic analogue of A in 0.1 will then be the ring of integers \mathbb{Z} or, more generally, rings of fractions A of rings of integers in an abelian extension of \mathbb{Q} . (The theory can be easily extended to the case when \mathbb{Q} is replaced by an arbitrary number field; we will not consider this more general situation in the book.) A prototypical example of such a ring is the ring

$$(0.2) \quad A = \mathbb{Z}[1/M, \zeta_N],$$

where M is an even integer, N is a positive integer, and ζ_N is a primitive N -th root of unity. Inverting M allows, as usual, to discard a set of “bad primes” (the divisors of M); adjoining ζ_N amounts to adjoining “new constants” to \mathbb{Z} .

Most of the times the ring A in 0.1 can be thought of as equal to $C^\infty(\mathbb{R}^m, \mathbb{R})$. But it will be sometimes useful to think of the ring A in 0.1 as consisting of analytic functions; indeed we will sometimes view analytic/algebraic functions as corresponding, in our theory, to global arithmetic objects in which case C^∞ objects will correspond, in our theory, to adelic objects. We will adopt these various viewpoints according to the various situations at hand.

The analogue of the set of coordinate functions

$$(0.3) \quad \mathcal{U} = \{\xi_1, \xi_2, \dots, \xi_m\} \subset C^\infty(\mathbb{R}^m, \mathbb{R})$$

on \mathbb{R}^m will be a (possibly infinite) set of primes,

$$(0.4) \quad \mathcal{V} = \{p_1, p_2, p_3, \dots\} \subset \mathbb{Z}$$

not dividing MN . We denote by $m = |\mathcal{V}| \leq \infty$ the cardinality of \mathcal{V} . (In case \mathbb{Q} is replaced by an arbitrary number field the set \mathcal{V} needs to be replaced by a set of finite places of that field.)

Next one considers the partial derivative operators on $A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$,

$$(0.5) \quad \delta_i := \delta_i^A : A \rightarrow A, \quad \delta_i f := \frac{\partial f}{\partial \xi_i}, \quad i \in \{1, \dots, m\}.$$

Following [23] we propose to take, as an analogue of the operators 0.5, the operators δ_p on $A = \mathbb{Z}[1/M, \zeta_N]$ defined by

$$(0.6) \quad \delta_p := \delta_p^A : A \rightarrow A, \quad \delta_p(a) = \frac{\phi_p(a) - a^p}{p}, \quad p \in \mathcal{V},$$

where $\phi_p := \phi_p^A : A \rightarrow A$ is the unique ring automorphism sending ζ_N into ζ_N^p .

More generally recall that a *derivation* on a ring B is an additive map $B \rightarrow B$ that satisfies the Leibniz rule. This concept has, as an arithmetic analogue, the concept of *p-derivation* defined as follows; cf. [23, 35, 83].

DEFINITION 0.1. Assume B is a ring and assume, for simplicity, that p is a non-zero divisor in B ; then a *p-derivation* on B is a set theoretic map

$$(0.7) \quad \delta_p := \delta_p^B : B \rightarrow B$$

with the property that the map

$$(0.8) \quad \phi_p := \phi_p^B : B \rightarrow B$$

defined by

$$(0.9) \quad \phi_p(b) := b^p + p\delta_p b$$

is a ring homomorphism.

We will always denote by ϕ_p the ring homomorphism 0.8 attached to a *p-derivation* δ_p as in 0.7 via the formula 0.9 and we shall refer to ϕ_p as the *Frobenius lift* attached to δ_p ; note that ϕ_p induces the *p-power Frobenius* on B/pB .

0.1.2. Classical differential geometry revisited. Classical differential geometric concepts are often introduced by considering frame bundles $P \rightarrow M$ of rank n vector bundles over m -dimensional manifolds M . For our purposes we take here $M = \mathbb{R}^m$. Such a P is a principal homogeneous space for the group $GL_n(\mathbb{R})$; and if the vector bundle is trivial (which, for simplicity, we assume in what follows) then P is identified with $M \times GL_n(\mathbb{R})$. Note that the rank n of the vector bundle and the dimension m of M in this picture are unrelated.

We want to review the classical concept of connection in P ; we shall do it in a somewhat non-standard way so that the transition to arithmetic becomes more transparent. Indeed consider an $n \times n$ matrix $x = (x_{ij})$ of indeterminates and consider the ring of polynomials in these n^2 indeterminates over some $A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$ as in 0.1, with the determinant inverted,

$$(0.10) \quad B = A[x, \det(x)^{-1}].$$

Note that B is naturally a subring of the ring $C^\infty(P, \mathbb{R})$. Denote by $\text{End}(B)$ the Lie ring of \mathbb{Z} -module endomorphisms of B . Also consider $G = GL_n = \text{Spec } B$ as a scheme over A .

DEFINITION 0.2. A *connection* on P (or on G , or on B) is a tuple $\delta = (\delta_i)$ of derivations

$$(0.11) \quad \delta_i := \delta_i^B : B \rightarrow B, \quad i \in \{1, \dots, m\},$$

extending the derivations 0.5. The *curvature* of δ is the matrix (φ_{ij}) of commutators $\varphi_{ij} \in \text{End}(B)$,

$$(0.12) \quad \varphi_{ij} := [\delta_i, \delta_j] : B \rightarrow B, \quad i, j \in \{1, \dots, m\}.$$

The *holonomy ring* of δ is the \mathbb{Z} -span \mathfrak{hol} in $\text{End}(B)$ of the commutators

$$(0.13) \quad [\delta_{i_1}, [\delta_{i_2}, [\dots, [\delta_{i_{s-1}}, \delta_{i_s}] \dots]]],$$

where $s \geq 2$; it is a Lie subring of $\text{End}(B)$.

In particular one can consider the *trivial connection* $\delta_0 = (\delta_{0i})$ defined by

$$(0.14) \quad \delta_{0i}x = 0.$$

Here $\delta_{0i}x = (\delta_{0i}x_{kl})$ is the matrix with entries $\delta_{0i}x_{kl}$; this, and similar, notation will be constantly used in the sequel. A connection is called *flat* if its curvature vanishes: $\varphi_{ij} = 0$ for all $i, j = 1, \dots, m$. For instance δ_0 is flat. For a flat δ the holonomy ring \mathfrak{hol} vanishes: $\mathfrak{hol} = 0$.

More generally let us define a *connection* on an arbitrary A -algebra B as a tuple of derivations 0.11 extending the derivations 0.5; one can then define the *curvature* by the same formula 0.12.

Note that, in classical differential geometry, a prominent role is played by connections on vector bundles; the framework of vector bundles is, however, too “linear” to have a useful arithmetic analogue: our arithmetic theory will be essentially a “non-linear” theory in which vector bundles (more generally modules) need to be replaced by principal bundles (more generally by rings).

There are various types of connections that we shall be interested in and for which we will seek arithmetic analogues; they will be referred to as:

- (1) Ehresmann connections,
- (2) Chern connections,
- (3) Levi-Civita connections,
- (4) Fedosov connections,
- (5) Lax connections,
- (6) Hamiltonian connections,
- (7) Cartan connections,
- (8) Riccati connections,
- (9) Weierstrass connections,
- (10) Painlevé connections.

In some cases our terminology above is not entirely standard. For instance what we call *Fedosov connection* is usually called *symplectic connection* [96, 62, 66]. We changed the name by which we refer to these connections in order to avoid confusion with the symplectic paradigm underlying the Hamiltonian case; our choice of name is based on the fact that symplectic manifolds equipped with a symplectic connection are usually called *Fedosov manifolds* [66]. Also there are a number of objects in classical differential geometry that are known under the name of *Cartan connection*; what we call *Cartan connection* is an object corresponding to what is usually called *Cartan distribution* [1], p. 133, and “lives” on jet bundles; our Cartan connection is unrelated, for instance, to the Maurer-Cartan connection. The Maurer-Cartan connection, on the other hand, will play a role in our book through the consideration of the *logarithmic derivative* map from an algebraic group to its Lie algebra. Finally what we call *connections* in cases (5), (6), (8), (9), (10) are classically referred to as *equations* or *flows*; e.g. what we call *Lax connection* is classically called *Lax equation*. Indeed connections classically refer to *linear* equations whereas the equations involved in (5)-(10) are *non-linear*. So what we call *connection* will not always refer to linear equations; this is in line with what will happen later in the arithmetic case where connections will be “intrinsically non-linear.”

In what follows we discuss the connections (1) through (7) in some detail. For these connections we assume A as in 0.1; for the connections (1) through (5) we assume, in addition, that B is as in 0.10; for connections (6) and (7) B will be defined when we get to discuss these connections. Connections (8), (9), (10) will not be discussed in this Introduction but will appear in the body of the book; these are connections on curves (for (8), (9)) or surfaces (for (10)) appearing in relation to the classical theory of differential equations with no movable singularities. The Painlevé connections (10) fit into the Hamiltonian paradigm (6). Both (9) and (10) lead to elliptic curves.

DEFINITION 0.3. A connection (δ_i) on B in 0.10 is an *Ehresmann connection* if it satisfies one of the following two equivalent conditions:

1a) There exist $n \times n$ matrices A_i with coefficients in A such that

$$(0.15) \quad \delta_i x = A_i x$$

1b) The following diagrams are commutative:

$$(0.16) \quad \begin{array}{ccc} B & \xrightarrow{\mu} & B \otimes_A B \\ \delta_i \downarrow & & \downarrow \delta_i \otimes 1 + 1 \otimes \delta_{0i} \\ B & \xrightarrow{\mu} & B \otimes B \end{array}$$

Here μ is the comultiplication. Condition 1a can be referred to as saying that (δ_i) is *right linear*. Condition 1b can be referred to as saying that δ is *right invariant*. (As we shall see the arithmetic analogues of conditions 1a and 1b will cease to be equivalent.) If (δ_i) is an Ehresmann connection the curvature satisfies $\varphi_{ij}(x) = F_{ij}x$ where F_{ij} is the matrix given by the classical formula

$$(0.17) \quad F_{ij} := \delta_i A_j - \delta_j A_i - [A_i, A_j].$$

Also there is a Galois theory attached to flat Ehresmann connections, the Picard-Vessiot theory. Indeed, for a flat Ehresmann connection $\delta = (\delta_i)$ consider the

logarithmic derivative map, $l\delta : GL_n(A) \rightarrow \mathfrak{gl}_n(A)^m$, with coordinates

$$(0.18) \quad l\delta_i(u) = \delta_i u \cdot u^{-1},$$

where \mathfrak{gl}_n is the Lie algebra of GL_n . The fibers of the map $l\delta$ are solution sets of systems of linear equations

$$(0.19) \quad \delta_i u = A_i \cdot u.$$

And if one replaces A by a ring of complex analytic functions then Galois groups can be classically attached to such systems; these groups are algebraic subgroups of $GL_n(\mathbb{C})$ measuring the algebraic relations among the solutions to the corresponding systems.

We next discuss Chern connections. We will introduce two types of Chern connections: *real Chern connections* and *complex Chern connections*. Complex Chern connections will be modeled on the classical hermitian connections on hermitian vector bundles [69, 88] which play a central role in complex geometry; real Chern connections are a real analogue of the complex ones and do not seem to play an important role in differential geometry (although see Duistermaat's paper [61] for a special case of these connections). Both real and complex Chern connections will have arithmetic analogues; the arithmetic analogue of real Chern connections will be more subtle (and hence more interesting) than the arithmetic analogue of the complex Chern connection so we will mostly concentrate on the former than on the latter.

To introduce these concepts let us consider again the rings A as in 0.1 and B as in 0.10, and let $q \in GL_n(A)$ be an $n \times n$ invertible matrix with coefficients in A which is either symmetric or anti-symmetric,

$$(0.20) \quad q^t = \pm q;$$

so the t superscript means here *transposition*. Of course, a symmetric q as above is viewed as defining a (semi-Riemannian) metric on the trivial vector bundle

$$(0.21) \quad M \times \mathbb{R}^n \rightarrow M,$$

on $M = \mathbb{R}^m$ while an anti-symmetric q is viewed as defining a 2-form on 0.21; here metrics/2-forms on vector bundles mean smoothly varying non-degenerate symmetric/anti-symmetric bilinear forms on the fibers of the bundle. With B be as in 0.10 let $G = GL_n = \text{Spec } B$ be viewed as a scheme over A and consider the maps of schemes over A ,

$$(0.22) \quad \mathcal{H}_q : G \rightarrow G, \quad \mathcal{B}_q : G \times G \rightarrow G,$$

defined on points by

$$(0.23) \quad \mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x, y) = x^t q y.$$

We continue to denote by the same letters the corresponding maps of rings $B \rightarrow B$ and $B \rightarrow B \otimes_A B$. Consider the *trivial* connection $\delta_0 = (\delta_{0i})$ on G defined by $\delta_{0i} x = 0$. Then one has the following easy results:

THEOREM 0.4. *There is a unique connection $\delta = (\delta_i)$ on G such that the following diagrams are commutative:*

$$(0.24) \quad \begin{array}{ccccc} B & \xleftarrow{\delta_i} & B & & B & \xleftarrow{\delta_{0i} \otimes 1 + 1 \otimes \delta_i} & B \otimes_A B \\ \mathcal{H}_q \uparrow & & \uparrow \mathcal{H}_q & & \delta_i \otimes 1 + 1 \otimes \delta_{0i} \uparrow & & \uparrow \mathcal{B}_q \\ B & \xleftarrow{\delta_{0i}} & B & & B \otimes_A B & \xleftarrow{\mathcal{B}_q} & B \end{array}$$

THEOREM 0.5. *There is a unique connection $\delta = (\delta_i)$ on G such that the following diagrams are commutative:*

$$(0.25) \quad \begin{array}{ccc} B & \xleftarrow{\delta_{0i} \otimes 1 + 1 \otimes \delta_i} & B \otimes_A B \\ \mathcal{H}_q \uparrow & & \uparrow \mathcal{B}_q \\ B & \xleftarrow{\delta_{0i}} & B. \end{array}$$

DEFINITION 0.6. A connection δ is \mathcal{H}_q -horizontal (respectively \mathcal{B}_q -symmetric) with respect to δ_0 if the left (respectively right) diagrams in 0.24 are commutative for all i . The unique connection that is \mathcal{H}_q -horizontal and \mathcal{B}_q -symmetric with respect to δ_0 (cf. Theorem 0.4) is called the *real Chern connection* attached to q .

DEFINITION 0.7. A connection δ is $(\mathcal{H}_q, \mathcal{B}_q)$ -horizontal with respect to δ_0 if the diagrams 0.25 are commutative for all i . The unique connection that is $(\mathcal{H}_q, \mathcal{B}_q)$ -horizontal with respect to δ_0 (cf. Theorem 0.5) is called the *complex Chern connection* attached to q .

Real and complex Chern connections turn out to be automatically Ehresmann connections. The arithmetic analogues of the real and complex Chern connections will not be ‘‘Ehresmann’’ in general.

The definitions just given may look non-standard. To see the analogy between our (real and complex) Chern connections and the classical hermitian Chern connections in [69, 88] we need to look at the explicit formulae for these connections; so we need to introduce more notation as follows.

For any connection (δ_i) on B in 0.10 set

$$(0.26) \quad \begin{aligned} A_i &= A_i(x) := \delta_i x \cdot x^{-1} \in \mathfrak{gl}_n(B), \\ \Gamma_{ij}^k &= -A_{ikj} = (k, j) \text{ entry of } (-A_i). \end{aligned}$$

The quantities Γ_{ij}^k will be referred to as *Christoffel symbols of the 2nd kind* of the connection. Also, for q as in 0.20, set

$$(0.27) \quad \Gamma_{ijk} := \Gamma_{ij}^l q_{lk},$$

which we refer to as the *Christoffel symbols of the 1st kind*. Here we use Einstein summation.

It is trivial to check that δ is \mathcal{H}_q -horizontal if and only if the following condition is satisfied:

$$(0.28) \quad \delta_i q_{jk} = \Gamma_{ijk} \pm \Gamma_{ikj}.$$

Classically if 0.28 holds one says that q is *parallel* with respect to δ .

Similarly it is trivial to check that δ is \mathcal{B}_q -symmetric if and only if the following condition is satisfied:

$$(0.29) \quad \Gamma_{ijk} = \pm \Gamma_{ikj}.$$

This is, as we shall see, a condition different from the classical condition of *symmetry* for a connection on the tangent bundle.

In 0.28 and 0.29 the upper (respectively lower) sign correspond to the upper (respectively lower) sign in 0.20.

Using 0.28 and 0.29 it is trivial to see that the real Chern connection attached to q exists and is unique, being given by

$$(0.30) \quad \Gamma_{ijk} = \frac{1}{2} \delta_i q_{jk}.$$

This proves Theorem 0.4. Note that, as promised earlier, the Chern connection is an Ehresmann connection, i.e. A_i belong to $\mathfrak{gl}_n(A)$ rather than $\mathfrak{gl}_n(B)$.

By the way the real Chern connection has the following compatibility with the special linear group SL_n : if $\delta_i(\det(q)) = 0$ then the real Chern connection $\delta_i : B \rightarrow B$ attached to q sends the ideal of SL_n into itself and hence induces a “connection on SL_n .” (This compatibility with SL_n will fail in the arithmetic case.)

Similarly it is easy to see that δ is $(\mathcal{H}_q, \mathcal{B}_q)$ -horizontal with respect to δ_0 if and only if

$$(0.31) \quad \Gamma_{ijk} = \delta_i q_{jk}.$$

Hence the complex Chern connection exists and is unique, being given by 0.31, which proves Theorem 0.5. Comparing 0.30 and 0.31 we see that the Christoffel symbols of the real and complex Chern connections coincide up to a $\frac{1}{2}$ factor and in this sense they are analogous to each other; on the other hand formula 0.31 is analogous to the formula for the classical hermitian Chern connection in [69, 88].

We next discuss Levi-Civita connections. Assume A as in 0.1, B as in 0.10, and assume that $n = m$. (Note we also implicitly assume here that a bijection is given between the set indexing the derivations δ_i and the set indexing the rows and columns of the $n \times n$ matrix $x = (x_{ij})$; such a bijection plays the role of what is classically called a *soldering*.) In the next definition we let z be an n -tuple of variables z_1, \dots, z_n .

DEFINITION 0.8. The connection (δ_i) is *symmetric* or *torsion free* if the following diagrams of A -algebras are commutative:

$$(0.32) \quad \begin{array}{ccc} B & \xleftarrow{s_i} & A[x] \\ s_j \uparrow & & \uparrow r_j \\ A[x] & \xleftarrow{r_i} & A[z] \end{array}$$

where $s_i(x) = -A_i^t$ and $r_j(z_k) = x_{jk}$.

Trivially, the commutativity of 0.32 is equivalent to:

$$(0.33) \quad \Gamma_{ijk} = \Gamma_{jik}.$$

Note the difference between the condition 0.33 defining *symmetry* and the condition 0.29 defining \mathcal{B}_q -*symmetry*: the two types of symmetry involve different pairs of indices. To avoid any confusion we will use the term *torsion free* rather than *symmetric* in what follows. The fundamental theorem of Riemannian geometry is, in this setting, the following (completely elementary) statement:

THEOREM 0.9. *Let $q \in GL_n(A)$, $q^t = q$. Then there is a unique connection δ that is \mathcal{H}_q -horizontal (i.e., satisfies equation 0.28 with the + sign) and is torsion free (i.e., it satisfies equation 0.33); it is given by*

$$(0.34) \quad \Gamma_{kij} = \frac{1}{2} (\delta_k q_{ij} + \delta_i q_{jk} - \delta_j q_{ki})$$

DEFINITION 0.10. The connection in Theorem 0.9 is called the *Levi-Civita* connection attached to q .

Note that, in particular, the Levi-Civita connection is an Ehresmann connection, i.e. A_i belong to $\mathfrak{gl}_n(A)$ rather than $\mathfrak{gl}_n(B)$. (The latter will fail in the

arithmetic case.) The Levi-Civita connection is, in a precise sense to be discussed later, “dual” to the real Chern connection and is generally different from the real Chern connection; it coincides with the real Chern connection if and only if

$$(0.35) \quad \delta_i q_{jk} = \delta_j q_{ik}$$

in which case q is called *Hessian* (the “real” analogue of *Kähler*).

The curvature of the Levi-Civita connection is, of course, the main object of Riemannian geometry. Let us recall some of the basic definitions related to it. First some notation. For (F_{ij}) the curvature 0.17 of the Levi-Civita connection attached to a metric $q = (q_{ij})$ we set:

$$(0.36) \quad F_{ij} = (F_{ijkl}), \quad R_{ij}^k := -F_{ijkl}, \quad R_{ijkl} = q_{im} R_{jkl}^m.$$

One refers to R_{ijkl} as the *covariant Riemann tensor*; the latter has following symmetries:

$$(0.37) \quad R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} + R_{iklj} + R_{ijlk} = 0, \quad R_{ijkl} = R_{klij}.$$

DEFINITION 0.11. A metric q is said to have *constant sectional curvature* if there exists $\kappa \in A$ with $\delta_i \kappa = 0$ such that

$$(0.38) \quad R_{ijkl} = \kappa \cdot (q_{ik} q_{jl} - q_{jk} q_{il}).$$

We next discuss Fedosov connections partially following [96, 62, 66, 119] where the term *symplectic* rather than *Fedosov* is being used. We consider, again, A as in 0.1 and B as in 0.10.

DEFINITION 0.12. Let $q \in GL_n(A)$, $q^t = -q$, so n is even. A *Fedosov connection* relative to q is a connection δ that is \mathcal{H}_q -horizontal (i.e., satisfies equation 0.28 with the $-$ sign) and is torsion free (i.e. it satisfies 0.33).

DEFINITION 0.13. $q \in GL_n(A)$, $q^t = -q$, is *symplectic* if it satisfies

$$(0.39) \quad \delta_i q_{jk} + \delta_j q_{ki} + \delta_k q_{ij} = 0.$$

One trivially checks that a Fedosov connection relative to q exists if and only if q is symplectic.

Unlike in [62, 66, 119] our Fedosov connections here are not necessarily Ehresmann. And, for a given symplectic matrix q , Fedosov connections that are Ehresmann exist but are not unique; one Fedosov connection which is an Ehresmann connection is given by

$$(0.40) \quad \Gamma_{ijk} = \frac{1}{3} (\delta_i q_{jk} + \delta_j q_{ik}).$$

We next discuss Lax connections. Let A be as in 0.1 and B as in 0.10.

DEFINITION 0.14. A connection (δ_i) on B is called a *Lax connection* if it satisfies

$$(0.41) \quad \delta_i x = [A_i(x), x] := A_i(x)x - xA_i(x)$$

for some $n \times n$ matrix $A_i(x)$ with coefficients in B .

Note that, unlike Chern and Levi-Civita connections, Lax connections are *not* a subclass of the Ehresmann connections. For a Lax connection, the following diagrams are commutative:

$$(0.42) \quad \begin{array}{ccc} B & \xleftarrow{\delta_i} & B \\ \mathcal{P} \uparrow & & \uparrow \mathcal{P} \\ A[z] & \xleftarrow{\delta_{0i}} & A[z] \end{array}$$

where $A[z] = A[z_1, \dots, z_n]$ is a ring of polynomials in the variables z_j , δ_{0i} are the unique derivations extending the corresponding derivations on A with $\delta_{0i}z_j = 0$, and \mathcal{P} is the A -algebra homomorphism with $\mathcal{P}(z_j) = \mathcal{P}_j(x)$,

$$(0.43) \quad \det(s \cdot 1 - x) = \sum_{j=0}^n (-1)^j \mathcal{P}_j(x) s^{n-j}.$$

The commutativity of 0.42 expresses the fact that the Lax connections describe “isospectral flows” on GL_n .

By the way the theory summarized above has a “(1,1)-analogue” for which we refer to the body of the book. Suffices to say here that, for the (1,1) theory, one may start with $M = \mathbb{C}^m$ and with a subring

$$(0.44) \quad A \subset C^\infty(M, \mathbb{C})$$

of the ring of smooth complex valued functions on M which is stable under the derivations

$$(0.45) \quad \delta_i := \frac{\partial}{\partial z_i}, \quad \delta_{\bar{i}} := \frac{\partial}{\partial \bar{z}_i}, \quad i = 1, \dots, m,$$

where z_1, \dots, z_m are the complex coordinates on M . A connection on $G = \text{Spec } B$, $B = A[x, \det(x)^{-1}]$, is then an m -tuple of derivations $\delta_i : B \rightarrow B$ extending the derivations $\delta_i : A \rightarrow A$. Consider the unique derivations $\delta_{\bar{i}} : B \rightarrow B$, extending the derivations $\delta_{\bar{i}} : A \rightarrow A$, such that $\delta_{\bar{i}}x = 0$. Then one defines the (1,1)-*curvature* of $\delta = (\delta_i)$ as the matrix $(\varphi_{i\bar{j}})$ with entries the A -derivations

$$(0.46) \quad \varphi_{i\bar{j}} := [\delta_i, \delta_{\bar{j}}] : B \rightarrow B.$$

The theory proceeds from here.

So far we discussed connections on the ring B in 0.10. In what follows we informally discuss Hamiltonian and Cartan connections; for a precise discussion we refer to the main text. These are connections on rings B other than 0.10.

To discuss Hamiltonian connections consider the ring

$$(0.47) \quad B = A[x],$$

with A as in 0.1 and $x = \{x_1, \dots, x_d\}$ a d -tuple of variables. Elements of the exterior products of the module of Kähler differentials $\Omega_{B/A}$ will be referred to as (vertical) *forms*. Also consider the derivations $\partial^j = \frac{\partial}{\partial x_j}$ on B .

DEFINITION 0.15. A 2-*form*

$$(0.48) \quad \omega := \omega^{ij} \cdot dx_i \wedge dx_j,$$

$\omega^{ij} \in B$, $\omega_{ij} + \omega_{ji} = 0$, is *symplectic* if $\det(\omega_{ij}) \in B^\times$ and

$$(0.49) \quad \partial^k \omega^{ij} + \partial^j \omega^{ki} + \partial^i \omega^{jk} = 0.$$

Note the contrast with the concept of *symplectic matrices* q : q are symplectic in the “horizontal directions,”

$$\delta_j = \partial/\partial\xi_j,$$

cf. 0.39, while 2-forms ω in 0.48 are symplectic in the “vertical directions,”

$$\partial^j = \frac{\partial}{\partial x_j}.$$

DEFINITION 0.16. For any 2-form ω as in 0.48 the *Lie derivative* $\delta_k\omega$ of ω with respect to ξ_k is defined by:

$$(0.50) \quad \delta_k\omega := (\delta_k\omega^{ij}) \cdot dx_i \wedge dx_j + \omega^{ij} \cdot d(\delta_k x_i) \wedge dx_j + \omega^{ij} \cdot dx_i \wedge d(\delta_k x_j).$$

DEFINITION 0.17. A *Hamiltonian* connection with respect to a symplectic 2-form ω is a connection on a B as in 0.47 whose Lie derivatives vanish:

$$\delta_k\omega = 0.$$

Hamiltonian connections with respect to symplectic forms naturally appear, by the way, in the background of some of the basic differential equations of mathematical physics, in particular in the background of the Painlevé VI equations. This kind of Hamiltonian connections turn out to have an arithmetic analogue [42] which will be discussed in the body of our book but not in this Introduction.

On the other hand one can consider *Hamiltonian connections* on a B as in 0.47 with respect to a *Poisson structure*. The Poisson story will be reviewed in the main body of the book. The two concepts of Hamiltonian connections (relative to symplectic forms and relative to Poisson structures) are related in case

$$\delta_k\omega^{ij} = 0.$$

An example of Poisson structure is provided by *Lie-Poisson structures* on Lie algebras in which case the corresponding Hamiltonian connections typically lead to Lax connections. A classical example of equations arising from a Lie-Poisson structure are the *Euler equations* for the rigid body which will be reviewed in the body of the book; the Euler equations are related to Lax equations in at least two different ways (via 3×3 and 2×2 matrices respectively). Euler equations will have an arithmetic analogue, although this arithmetic analogue will not be a priori related to our arithmetic analogues of Lax connections.

Finally we mention *Cartan connections*. In the same way we considered principal bundles and attached to them algebras B as in 0.10 one can consider “infinite jet bundles” (which we do not define here) and attach to them polynomial algebras,

$$(0.51) \quad B = A[x_j^{(\alpha)}; \alpha \in \mathbb{Z}_{\geq 0}^m, j = 1, \dots, d],$$

where A is as in 0.1 and $x_j^{(\alpha)}$ are indeterminates; these algebras come equipped with a natural flat connection

$$(0.52) \quad \delta_i := \frac{\partial}{\partial \xi_i} + \sum_j \sum_{\alpha} x_j^{(\alpha+e_i)} \frac{\partial}{\partial x_j^{(\alpha)}},$$

where e_i is the canonical basis of \mathbb{Z}^m .

DEFINITION 0.18. The flat connection (δ_i) on the ring B in 0.51 defined by 0.52 is called the *Cartan connection*.

Other names for our Cartan connection above are: the *total derivative* or the *Cartan distribution*. The Cartan connections have an arithmetic analogue which was thoroughly studied in [35, 34], will be reviewed in the main text of the present book in case $m = 1$, and plays a central role in the whole theory.

0.1.3. Arithmetic differential geometry. Roughly speaking arithmetic differential geometry is obtained from classical differential geometry by replacing classical differentiation (derivations) with arithmetic differentiation (p -derivations). This is often a convoluted process that we now explain.

The first step is to consider the ring

$$B = A[x, \det(x)^{-1}]$$

defined as in 0.10, with $x = (x_{ij})$ an $n \times n$ matrix of indeterminates, but where A is given now by

$$A = \mathbb{Z}[1/M, \zeta_N]$$

as in 0.2. We again consider the group scheme over A ,

$$G = GL_n := \text{Spec } B.$$

A first attempt to define arithmetic analogues of connections would be to consider families of p -derivations

$$\delta_p : B \rightarrow B, \quad p \in \mathcal{V},$$

extending the p -derivations 0.6; one would then proceed by considering their commutators on B (or, if necessary, expressions derived from these commutators). But the point is that the examples of “arithmetic analogues of connections” we will encounter in practice will essentially never lead to p -derivations $B \rightarrow B$! What we shall be led to is, rather, an adelic concept we next introduce. (Our guiding “principle” here is that, as mentioned before, C^∞ geometric objects should correspond to adelic objects in arithmetic while analytic/algebraic geometric objects correspond to global objects in arithmetic.)

To introduce our adelic concept let us consider, for each $p \in \mathcal{V}$, the p -adic completion of B :

$$(0.53) \quad B^{\widehat{p}} := \lim_{\leftarrow} B/p^n B.$$

Then we make the following:

DEFINITION 0.19. An *adelic connection* on $G = GL_n$ is a family $\delta = (\delta_p)$ of p -derivations

$$(0.54) \quad \delta_p := \delta_p^B : B^{\widehat{p}} \rightarrow B^{\widehat{p}}, \quad p \in \mathcal{V},$$

extending the p -derivations in 0.6.

If $\phi_p : B^{\widehat{p}} \rightarrow B^{\widehat{p}}$ are the Frobenius lifts attached to δ_p and $G^{\widehat{p}} = \text{Spf } B^{\widehat{p}}$ is the p -adic completion of $G = GL_n = \text{Spec } B$ then we still denote by $\phi_p : G^{\widehat{p}} \rightarrow G^{\widehat{p}}$ the induced morphisms of p -adic formal schemes.

Next we will explore analogues of the various types of connections encountered in classical differential geometry: Ehresmann, Chern, Levi-Civita, Fedosov, and Lax. (The stories of Hamiltonian connections and Cartan connections will be discussed in the main body of the book and will be skipped here.)

In what follows we need an analogue of the trivial connection in 0.14. It will be given by the adelic connection (δ_{0p}) defined by

$$(0.55) \quad \delta_{0p}x = 0.$$

The attached Frobenius lifts will be denoted by (ϕ_{0p}) ; they satisfy

$$(0.56) \quad \phi_{0p}(x) = x^{(p)}$$

where $x^{(p)}$ is the matrix (x_{ij}^p) . We call $\delta_0 = (\delta_{0p})$ the *trivial* adelic connection.

To introduce arithmetic analogues of Ehresmann connections one starts by noting that, for $n \geq 2$, there are *no* adelic connections $\delta = (\delta_p)$ whose attached Frobenius lifts (ϕ_p) make the following diagrams commute:

$$(0.57) \quad \begin{array}{ccc} G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \end{array}$$

Since 0.57 is an analogue of 0.16 one can view this as saying that there are no adelic connections that are analogues of right invariant connections. This is an elementary observation; we will in fact prove a less elementary result:

THEOREM 0.20. *For $n \geq 2$ and $p \nmid n$ there are no adelic connections (δ_p) and (δ_{1p}) whose attached Frobenius lifts (ϕ_p) and (ϕ_{1p}) make the following diagrams commute:*

$$(0.58) \quad \begin{array}{ccc} G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \\ \phi_p \times \phi_{1p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \end{array}$$

There is a useful property, weaker than the commutativity of 0.57, which we now introduce. Let T be the diagonal maximal torus of G and let N be the normalizer of T in G . We will say that an adelic connection (δ_p) with associated Frobenius lifts (ϕ_p) is *right invariant with respect to N* if the following diagrams are commutative:

$$(0.59) \quad \begin{array}{ccc} G^{\widehat{p}} \times N^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times N^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \end{array}$$

This latter property has its own merits but is too weak to function appropriately as a defining property of Ehresmann connections in arithmetic. Instead, we will consider an appropriate analogue of “linearity,” 0.15. What we will do will be to replace the Lie algebra \mathfrak{gl}_n by an arithmetic analogue of it, $\mathfrak{gl}_{n,\delta_p}$, and then we will introduce an arithmetic analogue of the logarithmic derivative. This new framework will naturally lead us to the following:

DEFINITION 0.21. An adelic connection (δ_p) is an *Ehresmann connection* if

$$(0.60) \quad \delta_p x = \alpha_p \cdot x^{(p)},$$

where α_p are matrices with coefficients in A .

By the way, clearly, Ehresmann connections are right invariant with respect to N . We will attach Galois groups to such Ehresmann connections and develop the basics of their theory. A natural expectation is that these Galois groups belong to the group $N(A)^\delta$ of all matrices in $N(A)$ whose entries are roots of unity or 0.

This expectation is not always realized but we will prove that something close to it is realized for (α_p) “sufficiently general.” The above expectation is justified by the fact that, according to the general philosophy of the field with one element \mathbb{F}_1 , the union of the $N(A)^\delta$'s, as A varies, plays the role of “ $GL_n(\mathbb{F}_1^a)$,” where \mathbb{F}_1^a is the “algebraic closure of \mathbb{F}_1 .”

Next we explain our arithmetic analogue of Chern connections.

Let $q \in GL_n(A)$ with $q^t = \pm q$. Attached to q we have, again, maps

$$\mathcal{H}_q : G \rightarrow G, \quad \mathcal{B}_q : G \times G \rightarrow G; \quad \mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x, y) = x^t q y.$$

We continue to denote by $\mathcal{H}_q, \mathcal{B}_q$ the maps induced on the p -adic completions $G^{\widehat{p}}$ and $G^{\widehat{p}} \times G^{\widehat{p}}$. Consider again the trivial adelic connection $\delta_0 = (\delta_{0p})$ on G (so $\delta_{0p}x = 0$) and denote by (ϕ_{0p}) the attached Frobenius lifts (so $\phi_{0p}(x) = x^{(p)}$). Then we will prove the following:

THEOREM 0.22. *For any $q \in GL_n(A)$ with $q^t = \pm q$ there exists a unique adelic connection $\delta = (\delta_p)$ whose attached Frobenius lifts (ϕ_p) make the following diagrams commute:*

$$(0.61) \quad \begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_p} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \xrightarrow{\phi_{0p}} & G^{\widehat{p}} \end{array} \quad \begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_{0p} \times \phi_p} & G^{\widehat{p}} \times G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \mathcal{B}_q \\ G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mathcal{B}_q} & G^{\widehat{p}} \end{array}$$

DEFINITION 0.23. An adelic connection $\delta = (\delta_p)$ is \mathcal{H}_q -horizontal (respectively \mathcal{B}_q -symmetric) with respect to $\delta_0 = (\delta_{0p})$ if the left (respectively right) diagrams in 0.61 are commutative. The unique connection which is \mathcal{H}_q -horizontal and \mathcal{B}_q -symmetric with respect to δ_0 (cf. Theorem 0.22) is called the *real Chern connection* attached to q .

The real Chern connection introduced above is an arithmetic analogue of the real Chern connection in classical differential geometry. Unlike in the case of classical differential geometry our adelic Chern connections will *not* be special cases of Ehresmann connections (although they *will* be right invariant with respect to N).

On the other hand we will have the following easy:

THEOREM 0.24. *For any $q \in GL_n(A)$ with $q^t = \pm q$ there is a unique adelic connection (δ_p) on GL_n whose attached Frobenius lifts (ϕ_p) make the following diagrams commute:*

$$(0.62) \quad \begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_{0p} \times \phi_p} & G^{\widehat{p}} \times G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{B}_q \\ G^{\widehat{p}} & \xrightarrow{\phi_{0p}} & G^{\widehat{p}} \end{array}$$

DEFINITION 0.25. An adelic connection $\delta = (\delta_p)$ is $(\mathcal{H}_q, \mathcal{B}_q)$ -horizontal with respect to $\delta_0 = (\delta_{0p})$ if the diagrams 0.62 are commutative. The unique connection which is $(\mathcal{H}_q, \mathcal{B}_q)$ -horizontal with respect to δ_0 (cf. Theorem 0.24) is called the *complex Chern connection* attached to q .

The concept of real Chern connection introduced above is, from an arithmetic point of view, more subtle than the concept of complex Chern connection. This can be seen already in the case of GL_1 . Indeed let

$$q \in GL_1(A) = A^\times, \quad A = \mathbb{Z}[1/M].$$

In this case it turns out that the ‘‘Christoffel symbols’’ defining the real Chern connection are related to the Legendre symbol; explicitly the Frobenius lift $\phi_p : G^{\widehat{p}} \rightarrow G^{\widehat{p}}$ corresponding to the real Chern connection attached to q is induced by the homomorphism

$$(0.63) \quad \phi_p : \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}} \rightarrow \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}}$$

satisfying

$$(0.64) \quad \phi_p(x) = q^{(p-1)/2} \left(\frac{q}{p} \right) x^p,$$

where $\left(\frac{q}{p} \right)$ is the Legendre symbol of $q \in A^\times \subset \mathbb{Z}_{(p)}$. On the other hand the Frobenius lift $\phi_p : G^{\widehat{p}} \rightarrow G^{\widehat{p}}$ corresponding to the complex Chern connection attached to q is simply defined by a homomorphism 0.63 satisfying

$$(0.65) \quad \phi_p(x) = q^{p-1} x^p.$$

In view of the above we will mainly concentrate, in this book, on the study of the real Chern connection while leaving the simpler case of the complex Chern connection to the reader. Note by the way that the quantity $\phi(x)/x^p$ in 0.65 is the square of the quantity $\phi(x)/x^p$ in 0.64; this has an analogue for GL_n with n arbitrary and is analogous to the fact that the Christoffel symbols of the classical complex Chern connection are twice the Christoffel symbols of the classical real Chern connection; cf. 0.30 and 0.31.

We shall also introduce, in our book, adelic connections that are analogues of Levi-Civita connections. They are already relevant in case \mathcal{V} consists of one prime only. So assume, in the theorem below, that $\mathcal{V} = \{p\}$ consists of one prime p . As usual we write $G = GL_n$ and we set $\mathfrak{g} = \mathfrak{gl}_n = \text{Spec } A[x]$.

THEOREM 0.26. *For any $q \in GL_n(A)$ with $q^t = q$ there is a unique n -tuple $(\delta_{1p}, \dots, \delta_{np})$ of adelic connections on G with attached Frobenius lifts*

$$(\phi_{1p}, \dots, \phi_{np}),$$

such that for all $i, j = 1, \dots, n$ the following diagrams are commutative

$$(0.66) \quad \begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{\phi_{ip}} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \xrightarrow{\phi_{0p}} & G^{\widehat{p}} \end{array} \quad \begin{array}{ccc} G^{\widehat{p}} & \xrightarrow{s_i} & \mathfrak{g}^{\widehat{p}} \\ s_j \downarrow & & \downarrow r_j \\ \mathfrak{g}^{\widehat{p}} & \xrightarrow{r_i} & (\mathbb{A}^n)^{\widehat{p}} \end{array}$$

where, for $\Delta_i := \delta_{ip}x$, the maps s_i, r_i are defined on points $a = (a_{ij}), b = (b_{ij})$ by:

$$s_i(a) = (a^{(p)})^{-1} \Delta_i(a), \quad r_i(b) = (b_{1i}, \dots, b_{ni}).$$

DEFINITION 0.27. The tuple $(\delta_{1p}, \dots, \delta_{np})$ in Theorem 0.26 is called the *Levi-Civita connection* attached to q ,

The commutativity of the left diagrams in 0.66 says, of course, that, for each i , (δ_{ip}) is \mathcal{H}_q -horizontal with respect to (δ_{0p}) and hence is analogous to the condition of parallelism 0.28 (with the + sign). The commutativity of the right diagrams in 0.66 is equivalent to the condition

$$(0.67) \quad \delta_{ip}x_{kj} = \delta_{jp}x_{ki}.$$

and is analogous to the condition of torsion freeness 0.33. This justifies our Definition 0.27 of the Levi-Civita connection. But note that, unlike in the case of

classical differential geometry, our adelic Levi-Civita connections are *not* special cases of Ehresmann connections.

The adelic Levi-Civita connection and the adelic real Chern connection attached to q will be related by certain congruences mod p that are reminiscent of the relation between the two connections in classical differential geometry.

The one prime paradigm of the Levi-Civita connection above can be viewed as corresponding to the classical Levi-Civita connections attached to (*translational*) *cohomogeneity one* metrics, by which we understand here metrics that, in appropriate coordinates, are invariant under a codimension one group of translations; equivalently metrics which in appropriate coordinates have the form $g = \sum g_{ij} d\xi_i d\xi_j$ with

$$(0.68) \quad \delta_k g_{ij} = \delta_l g_{ij}$$

for all i, j, k, l . As a rule we will omit, for simplicity, the adjective *translational* from now on.

Continuing to assume $\mathcal{V} = \{p\}$ one can also attempt to develop an arithmetic analogue of Fedosov connections as follows. Consider an anti-symmetric $q \in GL_n(A)$, $q^t = -q$. Let us say that an n -tuple of $(\delta_{1p}, \dots, \delta_{np})$ of adelic connections on $G = GL_n$ is a *Fedosov connection* relative to q if the diagrams 0.66 are commutative. We will prove that for $n = 2$ and any anti-symmetric q Fedosov connections relative to q exist. However, in contrast with the Levi-Civita story, we will prove that for $n \geq 4$ there is no Fedosov connection relative to the split q , for instance.

Finally there are adelic connections that are analogous to Lax connections. In fact there are two such analogues which we call *isospectral* and *isocharacteristic* Lax connections. They offer two rather different arithmetic analogues of isospectral flows in the space of matrices. Indeed the isocharacteristic property essentially says that a certain characteristic polynomial has “ δ -constant” coefficients whereas the isospectrality property essentially says that the characteristic polynomial has “ δ -constant” roots. (Here *δ -constant* means “killed by all δ_p ” which amounts to “being a root of unity or 0”.) In usual calculus the two properties are equivalent but in our arithmetic calculus these two properties are quite different. By the way isospectral and isocharacteristic Lax connections will not be defined on the whole of G but rather on certain Zariski open sets G^* and G^{**} of G respectively. Let us give some details of this in what follows. Let $G^* \subset G$ be the open set of *regular* matrices, consider the diagonal maximal torus T in G , and let $T^* = G^* \cap T$; *regular* means here “with distinct diagonal entries.” We will prove:

THEOREM 0.28. *There exists a unique adelic connection $\delta = (\delta_p)$ on G^* with attached Frobenius lifts (ϕ_p) such that the following diagrams commute:*

$$(0.69) \quad \begin{array}{ccc} (T^*)^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\phi_{0p} \times \phi_{0p}} & (T^*)^{\widehat{p}} \times G^{\widehat{p}} \\ \mathcal{C} \downarrow & & \downarrow \mathcal{C} \\ (G^*)^{\widehat{p}} & \xrightarrow{\phi_p} & (G^*)^{\widehat{p}}, \end{array}$$

where $\mathcal{C}(t, x) := x^{-1}tx$.

DEFINITION 0.29. The adelic connection δ in Theorem 0.28 is called the *canonical isospectral Lax connection*.

More generally let (δ_p) be the canonical isospectral Lax connection, with attached Frobenius lifts (ϕ_p) , $\phi_p(x) = \Phi_p$, i.e., $\Phi_p = (\Phi_{pij}(x))$ where $\Phi_{pij}(x) = \phi_p(x_{ij})$. Let $\alpha_p(x)$ be $n \times n$ matrices with entries in $\mathcal{O}(G^*)^{\widehat{p}}$. Then, setting $\epsilon_p(x) = 1 + p\alpha_p(x)$, one can consider the *isospectral Lax connection* attached to (α_p) , defined by the family of Frobenius lifts $(\phi_p^{(\epsilon)})$,

$$\phi_p^{(\epsilon)}(x) = \Phi_p^{(\epsilon)}(x) := \epsilon_p(x) \cdot \Phi_p(x) \cdot \epsilon_p(x)^{-1}.$$

The latter has the following property that justifies the term *isospectral*. Let

$$u_p = a_p^{-1} b_p a_p, \quad a_p \in GL_n(A^{\widehat{p}}), \quad b_p = \text{diag}(b_{p1}, \dots, b_{pn}) \in T^*(A^{\widehat{p}})$$

be such that $\phi_p^{(\epsilon)}(u) = \Phi_p^{(\epsilon)}(u)$. Then

$$\delta_p b_{pi} = 0, \quad i = 1, \dots, n.$$

So the eigenvalues b_{pi} of the “solutions” u_p are δ_p -constant.

On the other we will prove:

THEOREM 0.30. *There is an adelic connection $\delta = (\delta_p)$ on an open set $G^{**} \subset G$, $(G^{**})^{\widehat{p}} \neq \emptyset$, with attached Frobenius lifts (ϕ_p) , such that the following diagrams commute:*

$$(0.70) \quad \begin{array}{ccc} (G^{**})^{\widehat{p}} & \xrightarrow{\phi_p} & (G^{**})^{\widehat{p}} \\ \mathcal{P} \downarrow & & \downarrow \mathcal{P} \\ (\mathbb{A}^n)^{\widehat{p}} & \xrightarrow{\phi_{0p}} & (\mathbb{A}^n)^{\widehat{p}}, \end{array}$$

where $\mathbb{A}^n = \text{Spec } A[z]$.

The diagram 0.70 should be viewed as analogous to diagram 0.42. The adelic connection δ is, of course, not unique.

DEFINITION 0.31. An adelic connection δ as in Theorem 0.30 is called an *isocharacteristic Lax connection*.

Among isocharacteristic Lax connections there will be a *canonical* one.

For any isocharacteristic Lax connection (δ_p) , if $\phi_p(x) = \Phi_p(x)$ and $u_p \in G^{**}(A^{\widehat{p}})$ satisfies $\phi_p(u) = \Phi_p(u)$ we have that

$$\delta_p(\mathcal{P}_j(u)) = 0, \quad j = 1, \dots, n.$$

So the coefficients of the characteristic polynomials of the “solutions” u_p are δ_p -constant.

Note that the canonical Frobenius lifts ϕ_p in the two diagrams 0.69 and 0.70 will *not* coincide on the intersection $G^* \cap G^{**}$; so the isospectral and isocharacteristic stories are really different.

Next we would like to explore the curvature of adelic connections.

Consider first the case of Ehresmann connections, 0.60. Since $\alpha_p \in A$ for all p our Frobenius lifts $\phi_p : B^{\widehat{p}} \rightarrow B^{\widehat{p}}$ induce Frobenius lifts $\phi_p : A[x] \rightarrow A[x]$ and hence one can consider the “divided” commutators

$$(0.71) \quad \varphi_{pp'} := \frac{1}{pp'} [\phi_p, \phi_{p'}] : A[x] \rightarrow A[x], \quad p, p' \in \mathcal{V}.$$

The family $(\varphi_{pp'})$ will be referred to as the *curvature* of the Ehresmann connection (δ_p) .

The situation for general adelic connections (including the cases of real Chern and Lax connections) will be quite different. Indeed, in defining curvature we face the following dilemma: our p -derivations δ_p in 0.54 do not act on the same ring, so there is no a priori way of considering their commutators and, hence, it does not seem possible to define, in this way, the notion of curvature. It will turn out, however, that some of our adelic connections will satisfy a remarkable property which we call *globality along the identity* (more generally along various subvarieties); this property will allow us to define curvature via commutators.

DEFINITION 0.32. Consider the matrix $T = x - 1$, where 1 is the identity matrix. An adelic connection $\delta = (\delta_p)$ on GL_n , with attached family of Frobenius lifts (ϕ_p) , is *global along 1* if, for all p , $\phi_p : B^{\widehat{p}} \rightarrow B^{\widehat{p}}$ sends the ideal of 1 into itself and, moreover, the induced homomorphism $\phi_p : A^{\widehat{p}}[[T]] \rightarrow A^{\widehat{p}}[[T]]$ sends the ring $A[[T]]$ into itself. If the above holds then the *curvature* of (δ_p) is defined as the family of “divided” commutators $(\varphi_{pp'})$,

$$(0.72) \quad \varphi_{pp'} := \frac{1}{pp'} [\phi_p, \phi_{p'}] : A[[T]] \rightarrow A[[T]],$$

where $p, p' \in \mathcal{V}$. Let $\text{End}(A[[T]])$ denote the Lie ring of \mathbb{Z} -module endomorphisms of $A[[T]]$. Then define the *holonomy ring* \mathfrak{hol} of δ as the \mathbb{Z} -linear span in $\text{End}(A[[T]])$ of all the *Lie monomials*

$$[\phi_{p_1}, [\phi_{p_2}, \dots, [\phi_{p_{s-1}}, \phi_{p_s}] \dots]] : A[[T]] \rightarrow A[[T]]$$

where $s \geq 2$, $p_i \in \mathcal{V}$. Similarly define the *holonomy \mathbb{Q} -algebra* $\mathfrak{hol}_{\mathbb{Q}}$ of δ as the \mathbb{Q} -linear span of \mathfrak{hol} in $\text{End}(A[[T]]) \otimes \mathbb{Q}$. Finally define the *completed holonomy ring*,

$$\widehat{\mathfrak{hol}} = \varprojlim \mathfrak{hol}_n,$$

where \mathfrak{hol}_n is the image of the map

$$(0.73) \quad \mathfrak{hol} \rightarrow \text{End}(A[[T]]/(T)^n).$$

The various maps referred to above can be traced on the following diagram:

$$\begin{array}{ccccccccc} B^{\widehat{p}} & \subset & A^{\widehat{p}}[[T]] & \supset & A[[T]] & \subset & A^{\widehat{p}'}[[T]] & \supset & B^{\widehat{p}'} \\ \phi_p \downarrow & & \phi_p \downarrow & & \phi_p \downarrow \downarrow \phi_{p'} & & \downarrow \phi_{p'} & & \downarrow \phi_{p'} \\ B^{\widehat{p}} & \subset & A^{\widehat{p}}[[T]] & \supset & A[[T]] & \subset & A^{\widehat{p}'}[[T]] & \supset & B^{\widehat{p}'} \end{array}$$

The idea of comparing p -adic phenomena for different p 's by “moving along the identity section” was introduced in [34] where it was referred to as *analytic continuation between primes*. Analytic continuation is taken here in the sense of Zariski [124], Preface, pp. xii-xiii, who was the first to use completions of varieties along subvarieties as a substitute for classical analytic continuation over the complex numbers. The technique of analytic continuation is also used, in the form of *formal patching*, in inverse Galois theory [71]. But note that our use of analytic continuation is rather different from the one used in inverse Galois theory. Indeed, in our context, we are patching data defined on “tubular neighborhoods”

$$\text{Spf } B^{\widehat{p}} \quad \text{and} \quad \text{Spf } A[[T]]$$

of two *closed* subsets

$$\text{Spec } B/pB \quad \text{and} \quad \text{Spec } B/(T)$$

of the scheme $\text{Spec } B$, where the data are required to coincide on the “tubular neighborhood”

$$\text{Spf } A^{\widehat{p}}[[T]]$$

of the closed subset

$$\text{Spec } B/(p, T) = (\text{Spec } B/pB) \cap (\text{Spec } B/(T))$$

of $\text{Spec } B$. This is in contrast with the use of patching in Galois theory [71] where one patches data defined on two *open* sets covering a formal scheme.

Of course, the trivial adelic connection $\delta_0 = (\delta_{0p})$, $\delta_{0p}x = 0$, is global along 1 so it induces ring endomorphisms $\phi_{0p} : A[[T]] \rightarrow A[[T]]$,

$$\phi_{0p}(T) = (1 + T)^{(p)} - 1.$$

We may morally view δ_0 as an analogue of a flat connection in real geometry (where $A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$). Alternatively we may view δ_0 as an arithmetic analogue of the derivations $\delta_{\bar{z}_i} = \partial/\partial\bar{z}_i$ on $A[x, \det(x)^{-1}]$ which kill x , where $A \subset C^\infty(\mathbb{C}^m, \mathbb{C})$. Following this second analogy we may consider an arbitrary adelic connection $\delta = (\delta_p)$ global along 1, with attached Frobenius lifts (ϕ_p) , and introduce the following:

DEFINITION 0.33. The $(1, 1)$ -curvature of δ is the matrix of “divided commutators” $(\varphi_{p\bar{p}'})$

$$(0.74) \quad \varphi_{p\bar{p}'} := \frac{1}{pp'}[\phi_p, \phi_{0p'}] : A[[T]] \rightarrow A[[T]], \quad p \neq p',$$

$$(0.75) \quad \varphi_{p\bar{p}} := \frac{1}{p}[\phi_p, \phi_{0p}] : A[[T]] \rightarrow A[[T]].$$

Let us discuss next the curvature of real Chern connections. (The case of complex Chern connections is easier and will be skipped here.) We will prove:

THEOREM 0.34. *Let $q \in GL_n(A)$ with $q^t = \pm q$. If all the entries of q are roots of unity or 0 then the real Chern connection δ attached to q is global along 1; in particular δ has a well defined curvature and $(1, 1)$ -curvature.*

So we may address the question of computing the curvature and $(1, 1)$ -curvature of real Chern connections for various q 's whose entries are 0 or roots of unity. A special case of such q 's is given by the following:

DEFINITION 0.35. A matrix $q \in GL_n(A)$ is *split* if it is one of the following:

$$(0.76) \quad \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix},$$

where 1_r is the $r \times r$ identity matrix and $n = 2r, 2r, 2r + 1$ respectively.

We will prove:

THEOREM 0.36. *Let q be split and let $(\varphi_{pp'})$ and $(\varphi_{p\bar{p}'})$ be the curvature and the $(1, 1)$ -curvature of the real Chern connection on G attached to q . Then the following hold:*

- 1) Assume $n \geq 4$. Then for all $p \neq p'$ we have $\varphi_{pp'} \neq 0$.
- 2) Assume $n = 2r \geq 2$. Then for all p, p' we have $\varphi_{pp'}(T) \equiv 0 \pmod{(T)^3}$.
- 3) Assume $n = 2$ and $q^t = -q$. Then for all p, p' we have $\varphi_{pp'} = 0$.

- 4) Assume $n \geq 2$. Then for all p, p' we have $\varphi_{p\bar{p}'} \neq 0$.
 5) Assume $n = 1$. Then for all p, p' we have $\varphi_{pp'} = \varphi_{p\bar{p}'} = 0$.

Assertion 1 morally says that $\text{Spec } \mathbb{Z}$ is “curved,” while assertion 2 morally says that $\text{Spec } \mathbb{Z}$ is only “mildly curved.” Assertions 1 and 2 will imply, in particular, assertion 1 in the following:

THEOREM 0.37. *Assume q split and $n \geq 4$ is even. Then, for the real Chern connection attached to q , the following hold:*

- 1) $\widehat{\mathfrak{hol}}$ is non-zero and pronilpotent.
 2) $\mathfrak{hol}_{\mathbb{Q}}$ is not spanned over \mathbb{Q} by the components of the curvature.

Assertion 1 is in stark contrast with the fact, to be explained in the body of the book, that *holonomy Lie algebras* arising from Galois theory are never nilpotent unless they vanish. Assertion 2 should be viewed as a statement suggesting that the flavor of our arithmetic situation is rather different from that of classical *locally symmetric* spaces; indeed, for the latter, the Lie algebra of holonomy *is* spanned by the components of the curvature.

Note that the above theorem says nothing about the vanishing of the curvature $\varphi_{pp'}$ in case $n = 2, 3$ and $q^t = q$; our method of proof does not seem to apply to these cases.

By the way if $q \in GL_{2r}$ is symmetric and split then the real Chern connection $\delta_p : B^{\widehat{p}} \rightarrow B^{\widehat{p}}$ attached to q *does not* send the ideal of SL_n into itself; this is in contrast with the situation encountered in classical differential geometry. To remedy this situation we will construct connections on GL_n that *do* send the ideal of SL_n into itself; there are many such connections and the “simplest” one will be called the *special linear connection*.

Other curvatures can be introduced and vanishing/non-vanishing results for them will be proved; they will be referred to as *3-curvature*, *first Chern form*, *first Chern (1, 1)-form*, *mean curvature*, *scalar curvature*, etc. We will also introduce *real Chern connections* attached to hermitian metrics and results will be proved for their curvature. A similar theory can be developed for complex Chern connections.

Similar results will also be proved for the curvature of Lax connections. As we shall see the open sets G^* and G^{**} where isospectral and isocharacteristic Lax connections are defined (cf. 0.69 and 0.70) do not contain the identity of the group $G = GL_n$ hence curvature cannot be defined by analytic continuation along the identity; however these open sets will contain certain torsion points of the diagonal maximal torus of G and we will use analytic continuation along such torsion points to define curvature and (1, 1)-curvature. We will then prove the non-vanishing of the (1, 1)-curvature of isocharacteristic Lax connection for $n = 2$. The (canonical) isospectral Lax connection has, immediately from its definition, a vanishing curvature.

Note that the concept of curvature discussed above was based on what we called *analytic continuation between primes*; this was the key to making Frobenius lifts corresponding to different primes act on a same ring and note that it only works as stated for adelic connections that are global along 1. This restricts the applicability of our method to “metrics” q with components roots of unity or 0. One can generalize our method to include q 's with more general entries by replacing the condition of being global along 1 with the condition of “being global along

certain tori;” this will be discussed in the body of the book. However, there is a different approach towards making Frobenius lifts comparable; this approach is based on *algebraizing Frobenius lifts via correspondences* and works for adelic connections that are not necessarily global along 1 (or along a torus). The price to pay for allowing this generality is that endomorphisms (of $A[[T]]$) are replaced by correspondences (on GL_n). Let us explain this alternative road to curvature in what follows.

We will prove the following:

THEOREM 0.38. *Let $\delta = (\delta_p)$ be the real Chern connection on $G = GL_n$ attached to a matrix $q \in GL_n(A)$ with $q^t = \pm q$. Then there exists a family of diagrams of A -schemes*

$$(0.77) \quad \begin{array}{ccc} & Y_p & \\ \pi_p \swarrow & & \searrow \varphi_p \\ G & & G \end{array}$$

where π_p is affine and étale, inducing, via p -adic completion, a family of diagrams

$$(0.78) \quad \begin{array}{ccc} & Y_p^\widehat{p} & \\ \pi_p^\widehat{p} \swarrow & & \searrow \varphi_p^\widehat{p} \\ G^\widehat{p} & \xrightarrow{\phi_p} & G^\widehat{p} \end{array}$$

where $\pi_p^\widehat{p}$ are isomorphisms.

The triples

$$(0.79) \quad \Gamma_p = (Y_p, \pi_p, \varphi_p)$$

can be viewed as *correspondences* on G which provide “algebraizations” of our Frobenius lifts ϕ_p .

The family (Γ_p) is not uniquely attached to (δ_p) but does have some “uniqueness features” to be explained in the body of the book.

On the other hand if E is the field of rational functions on the scheme G and F_p is the finite E -algebra

$$F_p := \mathcal{O}(Y_p \otimes E)$$

then each correspondence Γ_p induces, via tensorization by E , a diagram

$$(0.80) \quad \begin{array}{ccc} & \text{Spec } F_p & \\ \pi_p \swarrow & & \searrow \varphi_p \\ \text{Spec } E & & \text{Spec } E \end{array}$$

The family of diagrams 0.80 will be referred to as a *correspondence structure* for (δ_p) . Note that each Γ_p induces an additive map $\Gamma_p^* : E \rightarrow E$,

$$(0.81) \quad \Gamma_p^*(z) := \text{Tr}_{\pi_p}(\varphi_p^*(z)), \quad z \in E,$$

where

$$\text{Tr}_{\pi_p} : F_p \rightarrow E$$

is the trace of the finite algebra map

$$\pi_p^* : E \rightarrow F_p$$

and

$$\varphi_p^* : E \rightarrow F_p$$

is induced by φ_p .

DEFINITION 0.39. The *curvature* of (Γ_p) is the matrix $(\varphi_{pp'}^*)$ with entries the additive homomorphisms

$$(0.82) \quad \varphi_{pp'}^* := \frac{1}{pp'} [\Gamma_p^*, \Gamma_{p'}^*] : E \rightarrow E, \quad p, p' \in \mathcal{V}.$$

Note that, in this way, we have defined a concept of curvature for real Chern connections attached to arbitrary q 's (that do not necessarily have entries zeroes or roots of unity). There is a $(1, 1)$ -version of the above as follows. Indeed the trivial adelic connection $\delta_0 = (\delta_{0p})$ has a canonical correspondence structure (Γ_{0p}) induced by

$$\Gamma_{0p} = (G, \pi_{0p}, \varphi_{0p}),$$

where π_{0p} is the identity, and $\varphi_{0p}(x) = x^{(p)}$.

DEFINITION 0.40. The $(1, 1)$ -*curvature* of (Γ_p) is the family $(\varphi_{p\bar{p}'}^*)$ where $\varphi_{p\bar{p}'}^*$ is the additive endomorphism

$$(0.83) \quad \varphi_{p\bar{p}'}^* := \frac{1}{pp'} [\Gamma_{0p'}^*, \Gamma_p^*] : E \rightarrow E \quad \text{for } p \neq p',$$

$$(0.84) \quad \varphi_{p\bar{p}}^* := \frac{1}{p} [\Gamma_{0p}^*, \Gamma_p^*] : E \rightarrow E.$$

Then we will prove the following:

THEOREM 0.41. *Let $q \in GL_2(A)$ be split. Then, for the real Chern connection attached to q , the following hold:*

- 1) *Assume $q^t = -q$. Then for all p, p' we have $\varphi_{pp'}^* = 0$ and $\varphi_{p\bar{p}'}^* \neq 0$.*
- 2) *Assume $q^t = q$. Then for all p, p' we have $\varphi_{p\bar{p}'}^* \neq 0$.*

Once again our results say nothing about curvature in case $n = 2$ and $q^t = q$; our method of proof does not seem to apply to this case.

Later in the body of the book the curvature and $(1, 1)$ -curvature of (Γ_p) introduced above will be called *upper *-curvatures* and we will also introduce the concepts of *lower *-curvature* and *lower *-(1, 1)-curvature* coming from actions on “cycles.”

Finally note that one can define curvature for the Levi-Civita connection.

DEFINITION 0.42. Let $(\delta_{1p}, \dots, \delta_{np})$ be the Levi-Civita connection attached to a symmetric $q \in GL_n(A)$. The *curvature* is defined as the family (φ_p^{ij}) , indexed by $i, j = 1, \dots, n$ given by the divided commutators

$$(0.85) \quad \varphi_p^{ij} := \frac{1}{p} [\phi_{ip}, \phi_{jp}] : \mathcal{O}(G^{\widehat{p}}) \rightarrow \mathcal{O}(G^{\widehat{p}}).$$

This is a “vertical” curvature (indexed by the index set of the columns and rows of x) rather than a “horizontal” curvature, in the style of the previously introduced curvatures (which are indexed by primes). We will prove non-vanishing results for these curvatures. For instance we have:

THEOREM 0.43. *Let i, j, k, l be fixed indices between 1 and n . Then, for the Levi-Civita connection, the following hold:*

- 1) *Assume $\delta_p q_{jk} + \delta_p q_{il} \not\equiv \delta_p q_{ik} + \delta_p q_{jl} \pmod{p}$. Then $\varphi_p^{ij}, \varphi_p^{kl} \not\equiv 0 \pmod{p}$.*
- 2) *Assume $n = 2r$ and q is split. Then $\varphi_p^{ij} \not\equiv 0 \pmod{p}$ for $i \neq j$.*

We will actually prove a more precise result than assertion 1 of the above Theorem. Indeed set

$$\Phi^{ij} := \Phi^{ij}(x) := \varphi^{ij}(x), \quad \Psi_{ij} := \Psi_{ij}(x) := x^{(p^2)^t} q^{(p^2)} \Phi^{ij}(x),$$

and let R_{ijkl} be the (k, l) -entry of the matrix Ψ_{ij} , so

$$(0.86) \quad \Psi_{ij} = (R_{ijkl}).$$

Then the R_{ijkl} in 0.86 can be viewed as an arithmetic analogue of the covariant Riemann tensor in classical differential geometry. We will prove that (R_{ijkl}) in 0.86 satisfies congruences mod p :

$$(0.87) \quad R_{ijkl} \equiv -R_{jikl}, \quad R_{ijkl} \equiv -R_{ijlk}, \quad R_{ijkl} + R_{iklj} + R_{ijlk} \equiv 0, \quad R_{ijkl} \equiv R_{klij},$$

which are, of course, an arithmetic analogue of 0.37. In addition we will prove the congruence

$$(0.88) \quad R_{ijkl} \equiv \frac{1}{2}(\delta q_{jk} + \delta q_{il} - \delta q_{ik} - \delta q_{jl})^p \pmod{(p, x-1)},$$

which can be viewed as an analogue of the condition of constant sectional curvature 0.38 for a metric that “degenerates into a square of a 1-form at the boundary;” the analogy will be explained in the body of the book.

Going back to arithmetic differential geometry, in case the entries of q are roots of unity or 0, we will prove that, for each i , the adelic connection (δ_{ip}) appearing in the Levi-Civita connection is global along 1 so we will be able to define a *mixed curvature* $(\varphi_{pp'}^{ij})$ indexed by $i, j = 1, \dots, n$ and $p, p' \in \mathcal{V}$ given by the divided commutators

$$(0.89) \quad \varphi_{pp'}^{ij} := \frac{1}{pp'} [\phi_{ip}, \phi_{jp'}] : A[[T]] \rightarrow A[[T]], \quad p \neq p',$$

$$(0.90) \quad \varphi_{pp}^{ij} := \frac{1}{p} [\phi_{ip}, \phi_{jp}] : A[[T]] \rightarrow A[[T]].$$

For a fixed p and the Fedosov connection $(\delta_{1p}, \delta_{2p})$ relative to any anti-symmetric $q \in GL_2(A)$ the formula 0.85 defines, again, a *curvature*; we will prove that this curvature does not vanish in general even if q is split.

0.2. Comparison with other theories

0.2.1. Analogies. From a number theoretic viewpoint there are at least two perspectives on the integers:

- A) Integers can be viewed as numbers.
- B) Integers can be viewed as analogous to functions.

The two perspectives sometimes coexist in various specific contexts in spite of the obvious tensions between them.

This being said, Perspective A is mainly the basis for subjects such as: the *geometry of numbers* [49]; *diophantine approximation* [50]; *p-adic analysis* [90]; *Fourier analysis* on adèles [94]; *p-adic mathematical physics* [120]; etc.

On the other hand perspective B interprets \mathbb{Z} as a ring of “functions” on a hypothetical “space;” cf. [102]. Here are some possibilities that have been suggested:

- B1) Integers are analogous to functions on a Riemann surface.
- B2) Integers are analogous to functions on a 3-dimensional manifold.
- B3) Integers are analogous to functions on an infinite dimensional manifold.

Accordingly the hypothetical “space” referred to above should be an analogue of a Riemann surface, a 3-manifold, or an infinite dimensional manifold, respectively.

Analogy B1 is classical, it has a complex geometric flavor, and is the basis of work stretching from Dedekind to Arakelov and beyond. According to this viewpoint the primes p in \mathbb{Z} are analogous to complex functions on a Riemann surface that vanish at one point only and hence they define points on the surface. A similar picture holds for integers in number fields. Note that Riemann surfaces are objects of complex dimension 1 (or real dimension 2). Genera of number fields are classically defined and finite, as in the case of Riemann surfaces. There is a related viewpoint according to which \mathbb{Z} is analogous to a ring of functions on a Riemann surface of infinite genus; cf., e.g., [55].

Analogy B2, in its topological form, originates in suggestions of Mazur, Manin, Kapranov, and others. According to this viewpoint the primes p in \mathbb{Z} correspond to embedded circles (knots) in a 3-dimensional manifold; so primes can be viewed as analogues of complex functions on the manifold that vanish along these knots. The Legendre symbol is then an analogue of linking numbers. Cf. Morishita’s monograph [104] and the references therein. A different form of analogy B2, with a dynamical flavor, was proposed by Deninger [59].

By contrast, the present book and previous work by the author adopt the analogy B3 and have a differential geometric flavor. The primes, in this setting, are analogous to global coordinate functions on an infinite dimensional manifold. Other viewpoints that fit into analogy B3 have also been proposed; cf. the discussion below on the field with one element.

There is a possibility that our theory has connections with viewpoints B1 and B2, as shown, for instance, by the presence of the Legendre symbol in our real Chern connections. Indeed the underlying Galois theory of reciprocity is an analogue of the monodromy in either the 2 or 3-dimensional picture; in the same way our arithmetic curvature theory could be an analogue of the identity component of a natural “holonomy” in one of these pictures.

There is also a possibility that our arithmetic differential geometry could play the role of framework for an *arithmetic physics* in the sense of Manin [100] which would then have perspective B in its background (as opposed to [120] which has perspective A in its background).

0.2.2. Field with one element. According to the viewpoint B described above integers can be viewed as functions on a hypothetical “space.” As noted, at least in its variant B1, this viewpoint is entirely classical. More recently this viewpoint was adopted in the search for a geometry over the *field with one element*, \mathbb{F}_1 ; cf. the pioneering suggestions of Tits [118]. See also the articles in [117], and the bibliography therein, for a survey of some approaches to the subject.

A remarkable idea appearing in the context of \mathbb{F}_1 is the consideration by Kurokawa et. al., [93] and Haran [70] of the *absolute derivation operators*,

$$\partial_p : \mathbb{Z} \rightarrow \mathbb{Z}, \quad \partial_p a := v_p(a) \frac{a}{p},$$

where $v_p(a)$ is the p -adic valuation of a . This is in line with paradigm B3. However the flavor of the operators ∂_p is rather different from that of the Fermat quotient operators δ_p and it seems unlikely that Haran’s theory and ours are directly related. Even more remote from our theory are the approaches to \mathbb{F}_1 in papers of Kapranov-Smirnov [86], Soulé [115], Deitmar [56, 117], Manin-Marcolli [117], Connes-Consani [55], etc., which do not directly provide an analogue of derivation for the integers.

Borger’s philosophy of \mathbb{F}_1 , cf. [11], is, in some sense, perpendicular to the above mentioned approaches to \mathbb{F}_1 . On the other hand, in the “case of one prime” Borger’s approach is consistent with ours: roughly speaking, in the case of one prime, Borger’s theory [11, 13] can be viewed as an algebraization of our analytic theory in [23, 35]. In the case of more (all) primes Borger’s \mathbb{F}_1 theory can also be viewed as a viewpoint consistent with B3 above: indeed Borger’s remarkable suggestion is to take λ -structures (in the sense of Grothendieck) as *descent data* from \mathbb{Z} to \mathbb{F}_1 . Recall that a λ -structure on a scheme X flat over \mathbb{Z} is the same as a commuting family (ϕ_p) of Frobenius lifts $\phi_p : X \rightarrow X$. So our theory would fit into “ λ -geometry” as long as:

- 1) the Frobenius lifts are defined on the schemes X themselves (rather than on the various p -adic completions $X^{\hat{p}}$) and
- 2) the Frobenius lifts commute.

However conditions 1 and 2 are almost never satisfied in our theory: the failure of condition 2 is precisely the origin of our curvature, while finding substitutes for condition 1 requires taking various convoluted paths (such as analytic continuation between primes or algebraization by correspondences). So in practice our approach places us, most of the times, outside the paradigm of λ -geometry.

0.2.3. Ihara’s differential. Next we would like to point out what we think is an important difference between our viewpoint here and the viewpoint proposed by Ihara in [78]. Our approach, in its simplest form, proposes to see the operator

$$\delta = \delta_p : \mathbb{Z} \rightarrow \mathbb{Z}, \quad a \mapsto \delta a = \frac{a - a^p}{p},$$

where p is a fixed prime, as an analogue of a derivation with respect to p . In [78] Ihara proposed to see the map

$$(0.91) \quad d : \mathbb{Z} \rightarrow \prod_p \mathbb{F}_p, \quad a \mapsto \left(\frac{a - a^p}{p} \bmod p \right)$$

as an analogue of differentiation for integers and he proposed a series of conjectures concerning the “zeroes” of the differential of an integer. These conjectures are still completely open; they are in the spirit of paradigm B1 listed above, in the sense that counting zeroes of 1-forms is a Riemann surface concept. But what we see as the main difference between Ihara’s viewpoint and ours is that *we do not consider the reduction mod p of the Fermat quotients but the Fermat quotients themselves*. This allows the possibility of considering compositions between our

δ_p 's which leads to the possibility of considering arithmetic analogues of differential equations, curvature, etc.

0.2.4. Fontaine-Colmez calculus. The Fontaine-Colmez theory of p -adic periods [64] also speaks of a *differential calculus with numbers*. Their calculus is perpendicular to ours in the following precise sense. For a fixed prime p our calculus based on the Fermat quotient operator δ_p should be viewed as a differential calculus in the “unramified direction,” i.e., the “direction” given by the extension

$$\mathbb{Q} \subset \bigcup_{p^t N} \mathbb{Q}(\zeta_N),$$

whereas the Fontaine-Colmez calculus should be viewed as a differential calculus in the “totally ramified direction,” i.e., roughly, in the “direction” given by the extension

$$\mathbb{Q} \subset \bigcup_n \mathbb{Q}(\zeta_{p^n}).$$

The Fontaine-Colmez theory is based on the *usual* Kähler differentials of totally ramified extensions and hence, unlike ours, it is about *usual* derivations. It is not unlikely, however, that, for a fixed p , a theory unifying the unramified and the totally ramified cases, involving two “perpendicular” directions, could be developed leading to arithmetic partial differential equations in two variables. Signs of possibility of such a theory can be found in our papers [32, 33]. This two variable setting would fit into paradigm B1 listed above but would be rather different from the classical forms (Dedekind style) of paradigm B1 discussed above.

0.2.5. Grothendieck’s p -curvature. It is worth pointing out that the study of our *curvature* here resembles the study of the p -*curvature* appearing in the arithmetic theory of differential equations that has been developed around the Grothendieck conjecture (cf., e.g., [87]). Both curvatures measure the lack of commutation of certain operators and both theories rely on technical matrix computations. However we should also point out that the nature of the above mentioned operators in the two theories is quite different. Indeed our curvature here involves the “ p -differentiation” of numbers with respect to primes p (in other words it is about d/dp and fits into paradigm B) whereas the theory in [87], and related papers, is about usual differentiation d/dt of power series in a variable t and essentially fits into paradigm A. In spite of these differences the two types of curvatures could interact; a model for such an interaction between d/dp and d/dt is in the papers [32, 33]. A similar remark can be made about the difference between our approach and that in [17] which, again, is about usual Kähler differentials, hence about usual derivations.

0.2.6. Discrete geometry as Euclidean geometry. Finally we would like to point out that the theory in this book is a priori unrelated to topics such as *the geometry of numbers* [49] on the one hand and *discrete differential geometry* [10] on the other. Indeed in both these geometries what is being studied are discrete configurations of points in the Euclidean space \mathbb{R}^m ; in the geometry of numbers the configurations of points typically represent rings of algebraic numbers (making the subject fit into paradigm A) while in discrete differential geometry the configurations of points approximate smooth submanifolds of the Euclidean space (making the subject a priori unrelated to number theory). This framework is, therefore,

that of the classical geometry of Euclidean space, based on \mathbb{R} -coordinates, and *not* that of an analogue of this geometry, based on “prime coordinates.” It may very well happen, however, that (one or both of) the above topics are a natural home for some (yet to be discovered) Archimedean counterpart of our (finite) adelic theory.

CHAPTER 1

Open problems

At this point *arithmetic differential geometry* is far from being a fully developed theory. What are then the next expected steps? And where is the theory headed? A number of puzzles, internal to the theory, as well as suggestive of connections with the rest of number theory, may be showing the right direction. We collect here a sample of such puzzles/problems, both technical and conceptual, that seem to indicate a general common tendency.

1.1. Unifying $\mathfrak{hol}_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$

One of the main objects of the theory presented in this book is the *holonomy algebra* $\mathfrak{hol}_{\mathbb{Q}}$ of an adelic connection; cf. Definition ???. (This is defined only for adelic connections that are global along appropriate subschemes; and as we saw this is the case with real Chern and Levi-Civita connections attached to large classes of metrics, in particular to δ -constant metrics.) We would like to view the Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$ as an infinitesimal analogue of the absolute Galois group $\Gamma_{\mathbb{Q}} = Gal(\mathbb{Q}^a/\mathbb{Q})$.

On the other hand if Hol is the holonomy group of a connection on principal G -bundle over a manifold M in classical differential geometry and if Hol^0 is its connected component then the quotient Hol/Hol^0 is usually referred to as the *monodromy group* of the connection; it is isomorphic to the image Γ_M of the monodromy representation

$$(1.1) \quad \pi_1(M) \rightarrow G$$

defined by our connection. (We have ignored here the base points.) The exact sequence

$$(1.2) \quad 1 \rightarrow Hol^0 \rightarrow Hol \rightarrow \Gamma_M \rightarrow 1$$

gives rise to a natural homomorphism

$$(1.3) \quad \Gamma_M \rightarrow \text{Out}(Hol^0).$$

Now it is classical to see the absolute Galois group $\Gamma_{\mathbb{Q}}$ as an arithmetic analogue of a fundamental group $\pi_1(M)$. Galois representations are then analogous to monodromy representations 1.1 and their images are analogous to the groups Γ_M . Since our Lie algebras $\mathfrak{hol}_{\mathbb{Q}}$ attached to real Chern or Levi-Civita adelic connections are an arithmetic analogue of the Lie algebra of Hol^0 and since we attached Lie algebras to Galois connections it is reasonable to pose the following:

Problem 1. “Unify” our holonomy on GL_n with Galois theory by constructing canonical extensions of Lie algebras

$$(1.4) \quad 0 \rightarrow \mathfrak{hol}_1 \rightarrow \mathfrak{hol}_2 \rightarrow \mathfrak{hol}_3 \rightarrow 0$$

where \mathfrak{hol}_3 is attached to a Galois connection and \mathfrak{hol}_1 is attached to a real Chern or Levi-Civita (adelic) connection on GL_n . Such an extension 1.4 should function as an arithmetic analogue of the extension 1.2 and the induced representation

$$(1.5) \quad \mathfrak{hol}_3 \rightarrow \text{Out}(\mathfrak{hol}_1)$$

should be an analogue of the representation 1.3. This construction could involve interesting Galois representations; indeed an extension such as 1.4 could be, in its turn, analogous to the basic exact sequence

$$(1.6) \quad 1 \rightarrow \pi_1(X^a) \rightarrow \pi_1(X) \rightarrow \Gamma_{\mathbb{Q}} \rightarrow 1$$

attached to a geometrically connected scheme X over \mathbb{Q} (where $X^a = X \otimes \mathbb{Q}^a$ and we suppressed, again, reference to base points); cf. [57, 77]. The representation 1.5 could then be analogous to the representation

$$(1.7) \quad \Gamma_{\mathbb{Q}} \rightarrow \text{Out}(\pi_1(X^a))$$

arising from 1.6. The representations 1.7 play a prominent role in work of Grothendieck, Ihara, Deligne, and many others.

Here are some related problems:

Problem 2. Find links between our curvature of adelic connections and reciprocity (as it appears in arithmetic topology [104]); the appearance of the Legendre symbol in our Christoffel symbols may be an indication that such links may exist.

Problem 3. Develop a Galois correspondence for Ehresmann connections; extend the Galois theory from Ehresmann connections to other types of connections, especially real Chern and special linear connections.

Problem 4. Find links between our adelic (flat) connections and Galois representations similar to the classical link between vector bundles with flat connections and monodromy.

Problem 5. Find an arithmetic analogue of Atiyah's theory [3] according to which:

- 1) the algebraic connections on an algebraic vector bundle over a variety are in bijection with the splittings of the Atiyah extension;
- 2) the obstruction to the splitting of the Atiyah extension generates the characteristic ring of the vector bundle.

Problem 6. The curvature of adelic Chern connections led, in our theory, to objects that we called *Chern (1,1)-forms*; it is tempting to use these forms, and higher versions of them, to define cohomology classes in the style of Chern-Weil theory; the target cohomology groups could be Galois (étale) cohomology groups of appropriate arithmetic objects.

1.2. Unifying “ $\partial/\partial p$ ” and “ $\partial/\partial \zeta_{p^\infty}$ ”

As discussed in the Introduction the theory of p -adic periods of Fontaine and Colmez involves, in particular, a “totally ramified arithmetic calculus” that is perpendicular to our (unramified) arithmetic calculus. In its simplest form this totally ramified calculus morally comes from the fact that the modules of Kähler differentials $\Omega_{\mathbb{Z}[\zeta_{p^n}]}$ are torsion but non-trivial: $d\zeta_{p^n} \neq 0$. It is conceivable that this situation could lead to a “derivation like” operator, “ $\partial/\partial \zeta_{p^\infty}$ ” on an appropriate space on which our p -derivation δ_p also acts. Then one is led to:

Problem 7. Unify our unramified arithmetic calculus (based on p -derivations δ_p , thought of as “ $\partial/\partial p$ ”) with the totally ramified arithmetic calculus of Fontaine-Colmez [64] (possibly involving the conjectural “derivation like” operators “ $\partial/\partial p^\infty$ ”). One would presumably get, for each p , a partial differential situation with two “directions” and, in particular, one could attempt to study the resulting “curvature.” A way to proceed could be to start from the paper [32] where the base ring was the ring of power series $R[[q]]$ in the variable q , equipped with the “standard” p -derivation δ_p (extending that of $R = W(\mathbb{F}_p^a)$ and killing q) and with the usual derivation $\delta_q = qd/dq$. Then one could try to specialize the theory in [32] via $q \mapsto \zeta_{p^n}$ with $n \rightarrow \infty$. Such a specialization does not a priori make sense, of course, and no recipe to give it a meaning seems available at this point. However an indication that this procedure might be meaningful can be found in the formulae giving the “explicit reciprocity laws” of local class field theory due to Shafarevich, Brückner, Vostokov, etc. Cf. [63].

1.3. Unifying Sh and GL_n

Problem 8. Relate the arithmetic differential calculus on Shimura varieties Sh (appearing in our book [35]) with the arithmetic differential calculus on the classical groups (appearing in our present book). In particular examine the case of

$$Sh = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO_2(\mathbb{R}).$$

An indication that such a relation may exist is the δ -algebraic theory of δ -Hodge structures in [22] giving δ -algebraic correspondences between classical groups and moduli spaces of abelian varieties; one would have to develop an arithmetic analogue of [22].

Here are some related questions:

Problem 9. Find a link between the concept of curvature in the present book and the concept of curvature based on the arithmetic Laplacian in [34]. More generally find a common ground between our theory here and [34].

Problem 10. The theory in this book is mainly about adelic connections on GL_n possibly compatible with involutions of GL_n . It is natural to explore what happens if one replaces GL_n by a general reductive group.

Problem 11. Explore the arithmetic differential geometry of homogeneous spaces such as “spheres,” $S^n := SO_{n+1}/SO_n$.

Problem 12. Develop an arithmetic analogue of the Cassidy-Kolchin theory of differential algebraic groups. Compute the rings of invariant δ -functions for actions of such groups on the varieties naturally appearing in Riemannian geometry. For instance compute the δ -functions on $SO_n \backslash GL_n / \Gamma$ where Γ is a δ -subgroup of GL_n . The δ -invariants of the curvature of the Levi-Civita connection may be playing a role in this problem.

Problem 13. Find interactions between the classical theory of quadratic forms over rings of integers in number fields [107] and our theory here (around symmetric matrices that play the role of metrics). Both theories are about the geometry of GL_n/SO_n .

1.4. Further concepts and computations

Problem 14. Decide if the curvature of the real Chern connection attached to a split symmetric $q \in GL_n$ vanishes for $n = 2, 3$. Compute (or prove non-vanishing of) curvature for the correspondence structures attached to real Chern connections on GL_n in case q is symmetric and $n = 2$. Same problem for curvature and $(1, 1)$ -curvature in case $n \geq 3$.

Problem 15. Prove vanishing/non-vanishing for the mixed curvature of the Levi-Civita connection attached to split q 's. Perform analytic continuation of Levi-Civita connection along partition tori and analyze the corresponding mixed curvature and gauge action.

Problem 16. Investigate the existence of correspondence structure for Levi-Civita connection; same question for the canonical isospectral and isocharacteristic Lax connections. In case such correspondence structures exist compute (or prove vanishing/non-vanishing of) their curvatures.

Problem 17. Study the impact of conformal changes of metrics on curvature.

Problem 18. Find the explicit structure of holonomy algebras $\mathfrak{hol}, \mathfrak{hol}^\Omega$ attached to real Chern connections.

Problem 19. Investigate the analytic properties of the Dirichlet series arising as our mean and scalar $(1, 1)$ -curvatures.

Problem 20. Further develop the formalism of curvature in the context of the ring of correspondences on a field.

Problem 21. Develop an arithmetic analogue of Ricci curvature, scalar curvature, and constant sectional curvature for the Levi-Civita connection. If this is to be done in the case of several (all) primes one will need a better understanding of the problem of soldering i.e., the problem of how to view GL_∞ as an analogue of the frame bundle of $Spec \mathbb{Z}$. On the other hand, in the case of one prime p , one should better understand the analogy between our formula for curvature mod p and the formula for the classical Riemann tensor of a metric with constant scalar curvature; cf. Remark ???. On a related note it would be important to develop an arithmetic analogue of covariant differentiation. This might require passing, in a systematic way, from adelic connections on GL_n to adelic connections on GL_N for any given rational representation (group scheme homomorphism) $\rho : GL_n \rightarrow GL_N$. Such a construction is, of course, available in classical differential geometry for Ehresmann connections. It would be desirable to have a similar construction in arithmetic differential geometry for Chern or Levi-Civita connections, at least in the case of representations ρ obtained from the tautological representation $id : GL_n \rightarrow GL_n$ by iterating the operations $\oplus, \otimes, \text{Hom}$.

Problem 22. Develop an arithmetic analogue of connections that are Hamiltonian with respect to the Lie-Poisson bracket. Find an arithmetic analogue of the theory of algebraically completely integrable systems [2], along the lines of the arithmetic analogue of the Euler equations for the rigid body discussed in the book; cf. [48]. Find a strengthening of the concept of Euler connection by requiring equality instead of congruence in the linearization condition.

Problem 23. Find an arithmetic analogue of Fedosov connections; furthermore find an arithmetic counterpart of deformation quantization. Indeed, the “strong” arithmetic analogue of Fedosov connections introduced in the text was proved not to exist in dimension ≥ 4 ; what is needed is the “correct weakening” of that concept.

Problem 24. Many of the curvature computations in our theory have the flavor of computations in *algebraic dynamics* (study of semigroups of endomorphisms of algebraic varieties); an instance of this is our use of ideas from [65] in the study of adelic Ehresmann connections. The connection with algebraic dynamics deserves further study.

Problem 25. Find an arithmetic analogue of geodesics. A hint could come from [32] where “time” was a geometric variable and “space” was represented by arithmetic variables (primes).

Problem 26. Perform curvature computations for complex Chern connections; as a rule they are easier than the corresponding computations for real Chern connections which were presented in the book. In particular complex Chern connections have obvious correspondence structure which is represented by field endomorphisms; and if attached to diagonal matrices they are always global along the diagonal maximal torus so Jordan multiplication ceases to play the role it played in the real case.

1.5. What lies at infinity?

The theory of the present book involves non-Archimedean (finite) adèles only. A natural question is whether there is an Archimedean (infinite) counterpart to our theory. So one can ask, for instance:

Problem 27. Could Borger’s positivity approach to study “Witt vectors at infinity” [15] interact with our theory? This sounds reasonable in view of the compatibility between our theory and Borger’s in the case of a single prime.

Problem 28. Could concepts of *discrete differential geometry* [10] play the role of Archimedean counterpart to our theory?

Problem 29. Is there an Archimedean counterpart to our theory that has the flavor of the classical *geometry of numbers* [49] or more generally of *Arakelov geometry*?

Problem 30. Could an *arithmetic physics* (in the sense of [100]) be developed, with our *arithmetic differential geometry* as background, that would have “real physics” as its Archimedean counterpart? This would offer a picture rather different from that in [120], as explained in the Introduction. Indeed the underlying analogy in [120] is between p -adic numbers and real numbers whereas the underlying analogy in our arithmetic differential geometry is between p -adic numbers and functions.

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