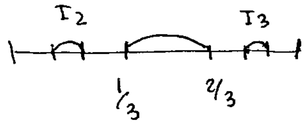
Fractal Dimension

(D) Let $K \subset \mathbb{R}$ be closed bounded, $\text{Conhull}(K) = [a,b]$, $\Omega = [a,b] \setminus K$,
$\Omega = \bigcup_{j=1}^{\infty} I_j$, $\text{length}(I_j) = \ell_j$, $\ell_1 \geq \ell_2 \geq \cdots > 0$. (So I exclude case of finitely many I_j's)

Minkowski dimension (fractal)(*) $D_M = \inf\{ s > 0 ;\ \underbrace{\sum_{j=1}^{\infty} \ell_j^s}_{} < \infty \}$ $=: \zeta_K(s)$. zeta fn of K

(EG) Cantor set K, def by $\Omega = \underbrace{(\tfrac{1}{3}, \tfrac{2}{3})}_{I_1} \cup I_2 \cup I_3 \cup \cdots$



$$\zeta_K(s) = \frac{1}{3^s} + \frac{2}{3^{2s}} + \frac{2^2}{3^{3s}} + \cdots = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} \ \cancel{= \frac{1}{3^s}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^s + \cdots\right)}$$

$$\cancel{= \frac{1}{3^s} \frac{1}{1 - \left(\frac{2}{3}\right)^s}}$$

~~So D_M = root of $\left(\frac{2}{3}\right)^s = 1$ $\Leftrightarrow$ $D_M = $~~ $=$

$$= \frac{1}{3^s}\left(1 + \frac{2}{3^s} + \left(\frac{2}{3^s}\right)^2 + \cdots\right) = \frac{1}{3^s} \frac{1}{1 - \frac{2}{3^s}}$$

D_M = root of $\frac{2}{3^s} = 1$ $\left(3^s = 2 \Rightarrow s = \log_3 2 = \frac{\log 2}{\log 3} \right)$

(*) not Hausdorff.