

# Galois ~~theory~~ Galois-style

Galois' original definition of his group.

(D)  $K$  a field,  $f \in K[x]$  a separable polynomial  
 $\alpha = (\alpha_1, \dots, \alpha_n) \in \bar{K}^n$  w/  $\{\alpha_1, \dots, \alpha_n\} = f^{-1}(0)$   
 $K(\alpha) = K(\alpha_1, \dots, \alpha_n)$ ,  $G_f = \text{Aut}(K(\alpha)/K)$  Galois gr of  $f$  (Artin's def)

$Q_f := \text{Ker}(K[x_1, \dots, x_n] \rightarrow K(\alpha), x_i \mapsto \alpha_i)$  ideal of relations among roots

(P)  $G_f$  nat iso to  $\Gamma_f = \left\{ \sigma \in S_n \mid \sigma Q_f = Q_f \right\} = \left\{ \sigma \in S_n \mid \begin{aligned} &F \in K[x_1, \dots, x_n] \text{ \& } F(\alpha_1, \dots, \alpha_n) = 0 \\ &\Rightarrow F(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) = 0 \end{aligned} \right\}$

Pf.  $\Gamma_f \xrightarrow{\text{nat}} G_f$  iso (EX) decomposition group  $A = K[s_1, \dots, s_n]$

(D)  $s_1, \dots, s_n \in K[x_1, \dots, x_n]$  fund sym polys,  $f = \sum c_i x^i$  monic,  $P_f = (s_1 + c_1, s_2 - c_2, \dots) \subset A$

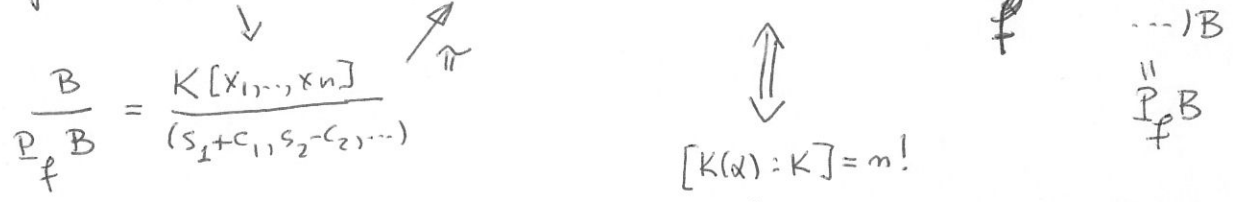
(R)  $A = K[s_1, \dots, s_n] \subset K[x_1, \dots, x_n] = \text{bis flat}$  (b/c  $K[s_1, \dots, s_n]$  is regular)  $\text{soloc free of rk } n!$

(b/c by Gal th  $K[s_1, \dots, s_n] \subset K[x_1, \dots, x_n]$  has deg  $n!$ ) Hence  $A/P_f \rightarrow B/P_f$  loc free of rk  $n!$

So  $\dim K[x_1, \dots, x_n]/(s_1 + c_1, s_2 - c_2, \dots) = n!$  if  $f = x^n + c_1 x^{n-1} + \dots + c_n$ .

(R)  $f$  sep  $\Leftrightarrow \text{Disc}^K := \prod (x_i - x_j)^2 \notin P_f$ .

(R) Have surj  $B = K[x_1, \dots, x_n] \rightarrow K(\alpha)$  - Hence  $\pi$  iso  $\Leftrightarrow Q_f = (s_1 + c_1, s_2 - c_2, \dots) \subset B$



Hence  $G_f = S_n \Leftrightarrow$  the id  $Q_f$  of rel's among roots is gen by the "Vieta" rels  $s_1 + c_1, s_2 - c_2, \dots$

(R)  $B \rightarrow \frac{B}{Q_f} = K(\alpha)$  induces  $K \text{ pt } f$   
 $B^{\Gamma_f} \rightarrow K(\alpha)^{G_f} = K$  which induces  $c: A \rightarrow A/P_f = K$ . So the latter is in the image  $(X/P_f)(K \rightarrow Y(K))$   
 So the set of all  $c$ 's w/ Gal gr  $\Gamma_f \subseteq S_n$  is "thin" (à la Serre)

(R)  $Q_f$  lies over  $P_f$ . If  $Q'_f$  is another max of  $B$  lying over  $P_f$  then the dec group  $\Gamma'_f = \{ \sigma \in S_n \mid \sigma Q'_f = Q'_f \}$  is conj to  $\Gamma_f$  in  $S_n$  (b/c  $S_n$  acts transitively on fibers of  $X = \text{Spec } B \rightarrow \text{Spec } A = Y$ ). So one can define (up to conj in  $S_n$ ) the Gal gr of  $f$  as the dec group of ANY maximal of  $B$  lying over  $P_f$ . This is a def that does not involve the roots!!!