Basic Galois Groups

① $\tilde{\rho} : G(\mathbb{F}_{p^n}/\mathbb{F}_p) \to \mathbb{Z}/n\mathbb{Z}$, $\sigma x = x^{p^{\tilde{\rho}(\sigma)}}$.

② $\rho : G(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \to (\mathbb{Z}/n\mathbb{Z})^\times$, $\sigma \zeta = \zeta^{\rho(\sigma)}$, $\zeta = \zeta^i$.

③ $\tilde{\rho} : G(K(\sqrt[n]{a})/K) \xrightarrow{\rho} \mu_n \simeq \mathbb{Z}/n\mathbb{Z}$, $\rho(\sigma) = \dfrac{\sigma \sqrt[n]{a}}{\sqrt[n]{a}} = \zeta_n^{\tilde{\rho}(\sigma)}$

$(\mu_n \subset K,\ \zeta_n \in \mu_n,\ (\sqrt[n]{a})^n = a)$

④ $\rho : G(\mathbb{Q}(\zeta_p, \sqrt[p]{a})/\mathbb{Q}) \to GL_2(\mathbb{F}_p)$, $\rho(\sigma) = \begin{bmatrix} \mu(\sigma) & \alpha(\sigma) \\ 0 & 1 \end{bmatrix}$

$\left[\begin{array}{l} \sigma \zeta_p = \zeta_p^{\mu(\sigma)}, \quad \sigma(\sqrt[p]{a}) = \zeta_n^{\alpha(\sigma)} \sqrt[p]{a}. \\ \operatorname{Im} \rho = \left\{ \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \right\}. \ (\text{degree} = p(p-1) \text{ b/c } (p, p-1) = 1) \end{array} \right]$

⑤ $\tilde{\rho} : G(K(\alpha)/K) \to \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$, $\tilde{\rho}(\sigma) = \sigma\alpha - \alpha$.

$[\ \alpha^p - \alpha = a \in K\ ]$