GAUGE FIELDS, MATTER FIELDS, GRAV. FIELDS

Alg. situation $k=\bar{k}$ alg cl field

M non sing var $/k$, G alg gr $/k$, $G \subset GL_n$ (closed), $\mathfrak{g} = Lie(G)$

V fin dim k-lin sp., $\rho: G \to GL(V)$ mor, $\rho' = ad\rho: \mathfrak{g} \to \mathfrak{gl}(V)$.

$\Omega^i(M)$ = i-forms on M, $G(M) = \mathrm{Hom}_k(M, G)$.

C^∞ situation (similar) (w/ V having a $\mathbb{C}$-str) Will wk in alg. sit for now but also in C^∞ sit when makes sense.

Note have $G \times V \xrightarrow{\rho} V$, $G \times \mathfrak{g} \xrightarrow{ad} \mathfrak{g}$ & map $\mathfrak{g} \times V \xrightarrow{\rho' = ad\rho} V$ is G-equiv

$g(A, v) = (gAg^{-1}, gv) \mapsto gAg^{-1}gv = gAv$

Let $U(\mathfrak{g}) = U$ be univ. envel. algebra. Define

(D) $[\,,\,] = \wedge_{\mathfrak{g}} : (\Omega^i(M) \otimes \mathfrak{g}) \times (\Omega^j(M) \otimes \mathfrak{g}) \to \Omega^{i+j} \otimes \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{id \times [\,,\,]} \Omega^{i+j} \otimes \mathfrak{g}$

$\wedge_U : \Omega^i(M) \otimes U(\mathfrak{g})) \times (\Omega^j(M) \otimes U(\mathfrak{g})) \to \Omega^{i+j} \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to \Omega^{i+j} \otimes U(\mathfrak{g})$

$\wedge_\rho : (\Omega^i(M) \otimes \mathfrak{g}) \times (\Omega^j(M) \otimes V) \to \Omega^{i+j} \otimes \mathfrak{g} \otimes V \xrightarrow{id \otimes \rho'} \Omega^{i+j} \otimes V$

(R) If $A = \sum \alpha_i \otimes \xi_i$, $B = \sum \beta_j \otimes \xi_j \Rightarrow A \wedge_U B = \sum (\alpha_i \wedge \beta_j) \otimes \xi_i \xi_j$, $[A, B] = \sum (\alpha_i \wedge \beta_j) \otimes [\xi_i, \xi_j]$

(D) $d: \Omega^i(M) \otimes \mathfrak{g} \to \Omega^{i+1}(M) \otimes \mathfrak{g}$, $d(A \wedge B) = dA \wedge B + (-1)^i A \wedge dB$
& sim for $U(\mathfrak{g})$ & V.

(Def) Elts of $\Omega^1(M) \otimes \mathfrak{g}$ called *gauge fields*

Group $G(M)$ called *gauge group*

Elts of $\Omega^i(M) \otimes V$ called *matter fields*.

$\Omega^1(M) \otimes \mathfrak{gl}_n(k) = \mathfrak{gl}_n(\Omega^1(M))$

(Prop) (Log. derivative) The map $G(M) \to \Omega^1(M) \otimes \mathfrak{gl}_n(k)$, $\gamma \mapsto \gamma^{-1} d\gamma$ takes values in $\Omega^1(M) \otimes \mathfrak{g}$. ($GL_n(\mathcal{O}(M)) \ni \gamma$, $\gamma^{-1}d\gamma \in \mathfrak{gl}_n(\Omega^1(M))$)

Pf Show $\forall\, \partial \in \mathrm{Der}_k(\mathcal{O}(M), \mathcal{O}(M))$, $\gamma^{-1}\partial\gamma \in \mathfrak{gl}_n(\mathcal{O}(M))$. Say $G = Z(F_\alpha)$.

So $F_\alpha(\gamma) = 0$ so $(1 + \varepsilon\partial)(F_\alpha(\gamma)) = 0$ in $k[\varepsilon]$

$\parallel$ Δ $1+\varepsilon\partial$ ring homo

$F_\alpha((1+\varepsilon\partial)(\gamma))$

$\parallel$

$F_\alpha(\gamma + \varepsilon\partial\gamma)$

$\Rightarrow \gamma + \varepsilon\partial\gamma \in G(k[\varepsilon])$, $\gamma^{-1} \in G(k[\varepsilon])$ $\Rightarrow \gamma^{-1}(\gamma + \varepsilon\partial\gamma) = I + \varepsilon\gamma^{-1}\partial\gamma \in G[\varepsilon]$ $\Rightarrow \gamma^{-1}\partial\gamma \in \mathfrak{g}(k)$ for each pt in M.

(Constr) Let $G(M) \times (\Omega^1(M) \otimes \mathfrak{g}) \to \Omega^1(M) \otimes \mathfrak{g}$

be action $(\gamma^{-1}, A) \mapsto A^{\gamma^{-1}} := \gamma^{-1}A\gamma + \gamma^{-1}d\gamma$ called gauge action

So gauge gr acts on gauge fields. $(ad\,\gamma^{-1})(A) = \rho'(\gamma^{-1})A$

(Rem) Also have action $G(M) \times (\Omega^i(M) \otimes \mathfrak{g}) \to \Omega^i(M) \otimes \mathfrak{g}$ by ad, $(\gamma, B) \mapsto \gamma B \gamma^{-1} = \rho'(\gamma) B$ (*)

(Def) Curvature of $A \in \Omega^1(M) \otimes \mathfrak{g}$ is $F = F_A := dA + A \wedge_U A$

$(A = \sum \alpha_i \otimes \xi_i \Rightarrow [A, A] = \sum (\alpha_i \wedge \alpha_j)(\xi_i\xi_j - \xi_j\xi_i) = 2A \wedge_U A) = dA + \frac{1}{2}[A, A] \in \Omega^2 \otimes \mathfrak{g}$

(Def) For $A \in \Omega^1(M) \otimes \mathfrak{g}$ def covariant exterior differentiation $D = D_A$

$$D\Phi = d\Phi + A \wedge_\rho \Phi, \quad D: \Omega^i(M) \otimes V \to \Omega^{i+1}(M) \otimes V.$$

(*) acting trivially on $\Omega^i(M)$ & simil have action $G(M) \times (\Omega^i(M) \otimes V) \to \Omega^i(M) \otimes V$ ind by ρ.

For $\Phi \in \Omega^i(M)\otimes V$

(P) $D_A^2 \Phi = F_A \wedge_\rho \Phi$ (in partic $D_A^2 : \Omega^i(M)\otimes V \to \Omega^{i+2}(M)\otimes V$ is $\mathcal{O}(M)$-linear)

Pf. $D^2\Phi = d(d\Phi + A\wedge_\rho \Phi) + A\wedge_\rho(d\Phi + A\wedge_\rho \Phi)$ $\quad (A\wedge_\cup A)\wedge_\rho \Phi$

$= 0 + dA\wedge_\rho \Phi - \cancel{A\wedge_\rho d\Phi} + \cancel{A\wedge_\rho d\Phi} + A\wedge_\rho(A\wedge_\rho \Phi) = F\wedge\Phi.$

(P) For $\Phi \in \Omega^i(M)\otimes V$ have $D_A(\gamma\Phi) = \gamma D_{A^{\gamma^{-1}}}\Phi$ (covariance of D)

Pf. $\gamma^{-1}D_A\gamma\Phi = \gamma^{-1}[d(\gamma\Phi) + A\wedge_\rho \gamma\Phi]$

$= \gamma^{-1}[d\gamma\wedge_\rho\Phi + \gamma d\Phi] + \gamma^{-1}[A\wedge_\rho\gamma\Phi]$

$= \gamma^{-1}d\gamma\wedge_\rho\Phi + d\Phi + \gamma^{-1}A\gamma\wedge_\rho\gamma^{-1}\gamma\Phi$

$\mathfrak{g}\times V \to V$ is G-equiv

$= d\Phi + (\underbrace{\gamma^{-1}d\gamma + \gamma^{-1}A\gamma}_{A^{\gamma^{-1}}})\wedge_\rho\Phi = D_{A^{\gamma^{-1}}}\Phi.$

(P) $F_{A^{\gamma^{-1}}} = \gamma^{-1}F_A\gamma$ (covariance of F)

Pf. Enough (check!) $F_{A^{\gamma^{-1}}}\wedge\Phi = \gamma^{-1}F_A\gamma\wedge_\rho\Phi$ for all ρ & Φ.

Now $F_{A^{\gamma^{-1}}}\wedge_\rho\Phi = D^2_{A^{\gamma^{-1}}}\Phi = D_{A^{\gamma^{-1}}}D_{A^{\gamma^{-1}}}\Phi = \gamma^{-1}D_A(\gamma\gamma^{-1}D_A(\gamma\Phi))$

$= \gamma^{-1}D_A^2(\gamma\Phi) = \gamma^{-1}(F_A\wedge_\rho\gamma\Phi) = (\gamma^{-1}F_A\gamma)\wedge_\rho\Phi$ $\square$.

(P) (Bianchi id) Look @ case $V=\mathfrak{g}$, $\rho = \mathrm{ad}$. So $\left\{ \begin{array}{l} A\wedge_\rho B = [A,B] \\ = A\wedge_\cup B - B\wedge_\cup A \end{array}\right.$

for $A \in \Omega^1(M)\otimes\mathfrak{g}$, $B\in\Omega^2(M)\otimes\mathfrak{g}$. Then $\left\{ \begin{array}{l} dF_A = [F_A, A] \\ D_A F_A = 0 \end{array}\right.$

Pf. $dF_A = d(dA + A\wedge_\cup A) = \underbrace{d^2A}_{0} + dA\wedge_\cup A - A\wedge_\cup dA$

$= (F - A\wedge_\cup A)\wedge_\cup A - A\wedge_\cup(F - A\wedge_\cup A)$

$= F\wedge_\cup A - A\wedge_\cup F = [F,A] = -[A,F] = -A\wedge_\rho F$. So $D_A F_A = 0$

(D) $G(M)$ acts on $[\underbrace{\mathcal{O}(M)\otimes V}_{\substack{\text{matter fields} \\ \mathcal{F}_{mat}}}] \times [\underbrace{\Omega^1(M)\otimes\mathfrak{g}}_{\substack{\text{gauge fields} \\ \mathcal{F}_{gauge}}}]$ (ρ on 1st factor, gauge act on 2nd) $\quad (\gamma, (\Phi, A)) \mapsto (\gamma\Phi, A^\gamma)$

A "diff-alg" fcn $L : \mathcal{F}_{mat}\times\mathcal{F}_{gauge} \to \Omega^d(M)$ is called a Lagrangian ($d = \dim M$). L called gauge inv if L const on $G(M)$-orbits.

(& $*$ induced by sheaf iso $\Omega^i_M \simeq \Omega^{d-i}_M$)

(Def) An k-lin op $* : \bigoplus_{i=0}^{d} \Omega^i(M) \circlearrowleft$ called a Hodge op if bij & $*\Omega^i(M) = \Omega^{d-i}(M)$.
It induces $G(M)$-equiv bij $*: \Omega^i(M)\otimes \mathfrak{g} \to \Omega^{d-i}(M)\otimes\mathfrak{g}$ where $G(M)$ acts on $\Omega^i(M)\otimes\mathfrak{g}$ via ad (& sim for $\mathfrak{g}$ repl by V)

(EG) (examples of gauge inv Lagr's) $\dim M = d = 4$

1) $L_1(\Phi, A) := \mathrm{tr}(F_A \wedge_{\mathfrak{g}} * F_A)$ (where $\mathrm{tr}: \Omega^4(M)\otimes\mathfrak{g} \xrightarrow{\mathrm{id}\otimes\mathrm{tr}} \Omega^4(M)$, $\mathrm{tr}: \mathfrak{g} \subset \mathfrak{gl}_n(k) \xrightarrow{\mathrm{Tr}} k$)

self int of A

Indeed $L_1(\gamma^{-1}\Phi, A^{\gamma^{-1}}) = \mathrm{tr}(F_{A^{\gamma^{-1}}} \wedge_{\mathfrak{g}} * F_{A^{\gamma^{-1}}}) = \mathrm{tr}(\gamma^{-1}F_A\gamma \wedge_{\mathfrak{g}} * \gamma^{-1}F_A\gamma)$
$= \mathrm{tr}(\gamma^{-1}F_A \underbrace{\gamma \wedge_{\mathfrak{g}} \gamma^{-1}} * F_A \gamma) = \mathrm{tr}(\gamma^{-1}F_A \wedge_{\mathfrak{g}} * F_A\gamma)$
$= \mathrm{tr}(F_A \wedge_{\mathfrak{g}} * F_A)$. ($\mathrm{id}$)

2) $L_2(\Phi, A) = \Phi^t \wedge * \Phi$ (where $V = \mathrm{Mat}_{n,1} \ni \Phi$ & $\Phi^t \in \mathrm{Mat}_{1,n} =: V^t$ is the transpose w/ $G \subset O(n)$ acting on V & V' naturally (*)).

self int of Φ

Indeed $(\gamma\Phi)^t \wedge * \gamma\Phi = \Phi^t\gamma^t \wedge * \gamma\Phi = \Phi^t \wedge * \Phi$.
b/c $*\gamma = \gamma *$ & $\gamma^t\gamma = \mathrm{id}$

3) $L_3(\Phi, A) = (D_A\Phi)^t \wedge * D_A\Phi$ (again $G \subset O(n)$, $V = \mathrm{Mat}_{n,1}$ etc as in 2)) (*)

Inter of Φ w/ A

Indeed $(D_{A\gamma}(\gamma\Phi))^t \wedge * D_{A\gamma}(\gamma\Phi) = (D_A\Phi)^t \gamma^t \wedge \gamma * D_A\Phi = (D_A\Phi)^t \wedge * D_A\Phi$.
$D_{A\gamma}\gamma\Phi = \gamma D_A\Phi$

4) $L_{YM} = L_1 + L_2 + L_3$ (Yang Mills Lagrangian, in the $U(n)$ version)

(D) Let E be a vect bdle on M. A connection on E is a k-lin map $\nabla : E \to E\otimes\Omega^1_M$ s.t. $\nabla(fe) = e\,df + f\nabla e$. $\exists!$ $\nabla: E\otimes\Omega^\bullet_M \circlearrowleft$ s.t. ∇ is the old ∇ on E & $\nabla(e\otimes \underset{\Omega^i}{w}) = \nabla e \wedge w + e\otimes dw$. This has also prop $\nabla(\underset{E\otimes\Omega^i}{\eta}\wedge \underset{\Omega^j}{w}) = \nabla\eta\wedge w + (-1)^i \eta\wedge dw$.
It foll that $\nabla^2(f\underset{E}{e}) = \nabla(e\otimes df + \nabla e\otimes f) = \cancel{\nabla e\otimes df} + \nabla^2 e\otimes f - \cancel{\nabla e\otimes df}$ so $\nabla^2: E \to E\otimes\Omega^2$ is $\mathcal{O}$-linear so it def an elt of $H^0((\mathrm{End}\,E)\otimes_{\mathcal{O}}\Omega^2) = \mathfrak{gl}_n(k)\otimes_k \Omega^2(M)$ called F_∇ (curv of ∇)
if E trivial $\simeq \mathcal{O}^n$

(This suggests to see connections on (non-triv) vect bdles (or bdles w/ other grs than GL_n) as (generalisations of) gauge fields. Sections in such bdles are (generalized) matter fields)

(P) In not's above, assume E trivial ($= \mathcal{O}\otimes V$, $G = GL_n \circlearrowright V = k^n$). Then 1) $\underbrace{\mathcal{O}(M)\otimes V}_{\text{matter fields}} = \mathcal{O}(E) = \{\text{sect'ns of } E\}$

2) $\underbrace{\dfrac{\Omega^1(M)\otimes\mathfrak{gl}_n(k)}{GL_n(M)}}_{\text{gauge fields mod gauge equiv}} \simeq \{\text{connections on } E\}$ & curvatures correspond

Pf If ∇ conn & (e_i) basis write $\nabla e_i = \sum e_j\otimes w_{ij}$ & attach $(w_{ij})^t$ mod gauge eq.

(*) $\exists$ obvious version with w/ $O(n)$ repl by $U(n)$ in which $(\ )^t$ repl by $\overline{(\ )}^t$. (in C^∞ case)

Set $T = T_M$, $\Omega = \Omega^1_M$

(D) A gravit field is an elt $g \in \overbrace{(\Omega\otimes\Omega)^{S_2}_0}^{\mathcal{F}_{grav}} \subset \Omega\otimes\Omega$ (also called metric)
where "index 0" means "non-deg". $\qquad \cong (S^2\Omega)_0$

If ω^α basis of Ω, $g = g_{\alpha\beta}\,\omega^\alpha\otimes\omega^\beta$ (Einst not'n, $g_{\alpha\beta} = g_{\beta\alpha}$) non-deg means $\det g := \det g_{\alpha\beta} \in \mathcal{O}^\times$

(D) A connection on T is (cf gen case above) a k-lin map $\nabla : T \to T\otimes\Omega$ w/ Leibn rule.
If ω^α as above & ∂_α basis of T dual to ω^α (i.e. $\partial_\alpha : \mathcal{O}\to\mathcal{O}$ k-der, $\langle \partial_\alpha, \omega^\beta\rangle = \delta^\alpha_\beta$)
then $\nabla\partial_\mu = \underbrace{\Gamma^\nu_{\mu\sigma}}_{\text{"Cristoffel symb's"}}\,\partial_\nu\otimes\omega^\sigma$, $\quad d\underset{\in \mathcal{O}}{f} = \partial_\mu f\cdot\omega^\mu$

so $\nabla(V^\mu\partial_\mu) = \partial_\mu\otimes dV^\mu + \nabla\partial_\mu\cdot V^\mu = \partial_\mu\otimes\partial_\sigma V^\mu\,\omega^\sigma + \Gamma^\nu_{\mu\sigma}V^\mu\,\partial_\nu\otimes\omega^\sigma$

$= \underbrace{\left[\partial_\sigma V^\nu + \Gamma^\nu_{\mu\sigma}V^\mu\right]}_{=:\ \nabla_\sigma V^\nu}\partial_\nu\otimes\omega^\sigma$

(R) Have natural $\otimes$ & dual of vect bdles w/ conn's s.t. $\nabla(\underset{\in E}{e}\otimes\underset{\in F}{f}) = \nabla e\otimes f + e\otimes\nabla f$ (def via $E\otimes\Omega\otimes F \cong E\otimes F\otimes\Omega$), $\quad d\langle \underset{\in E}{e}, \underset{\in \check{E}}{\check{e}}\rangle = \langle\nabla e, \check{e}\rangle + \langle e, \nabla\check{e}\rangle$

In partic ∇ on T induces ∇ on Ω & on $T\otimes\cdots\otimes T\otimes\Omega\otimes\cdots\otimes\Omega = T^{\otimes p}\otimes\Omega^{\otimes q}$

For $\tau\in T^{\otimes p}\otimes\Omega^{\otimes q}$ write $\tau = \tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}\,\partial_\alpha\otimes\partial_\beta\otimes\cdots\otimes\omega^{\alpha'}\otimes\omega^{\beta'}\otimes\cdots$

Image of τ via $T^{\otimes p}\otimes\Omega^{\otimes q}\xrightarrow{\text{cont}} T^{\otimes p-1}\otimes\Omega^{\otimes q-1}$ induced by $T\otimes\Omega\to\mathcal{O}$ (on 1st place) is $(\underbrace{\tau^{\alpha\beta\gamma\cdots}_{\alpha\beta'\gamma'\cdots}}_{\text{Einst summ}})\,\partial_\beta\otimes\partial_\gamma\otimes\cdots\otimes\omega^{\beta'}\otimes\omega^{\gamma'}\otimes\cdots$

Note $\nabla\omega^\nu = -\Gamma^\nu_{\mu\sigma}\,\omega^\mu\otimes\omega^\sigma$ (b/c $0 = d\langle\partial_\mu,\omega^\nu\rangle = \langle\underbrace{\nabla\partial_\mu}_{=\Gamma^\rho_{\mu\sigma}\partial_\rho\otimes\omega^\sigma}, \omega^\nu\rangle + \langle\partial_\mu, \underbrace{\nabla\omega^\nu}_{?^\nu_{\rho\sigma}\omega^\rho\otimes\omega^\sigma}\rangle$)

$\qquad\qquad \Gamma^\nu_{\mu\sigma}\,\omega^\sigma \qquad ?^\nu_{\mu\sigma}\,\omega^\sigma$

More generally $\nabla(\tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}\,\partial_\alpha\otimes\cdots\otimes\omega^{\alpha'}\otimes\cdots) = \longleftrightarrow \quad \partial_\alpha\otimes\cdots\otimes\omega^{\alpha'}\otimes\cdots\otimes d\tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}$

$+\ \tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}\,\nabla\partial_\alpha\otimes\cdots\otimes\omega^{\alpha'}\otimes\cdots + \tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}\,\partial_\alpha\otimes\cdots\otimes\nabla\omega^{\alpha'}\otimes\cdots$

$(*)$
$$
\begin{aligned}
&= \partial_\sigma\tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}\cdot\partial_\alpha\otimes\cdots\otimes\omega^{\alpha'}\otimes\cdots\otimes\omega^\sigma + \tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}\,\Gamma^\nu_{\alpha\sigma}\,\partial_\nu\otimes\cdots\otimes\omega^{\alpha'}\otimes\cdots\otimes\omega^\sigma\ \Big|_{\substack{\nu\mapsto\alpha\\ \alpha\mapsto\nu}}\\
&\quad\cdots - \tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}\,\Gamma^{\alpha'}_{\mu'\sigma'}\,\partial_\alpha\otimes\cdots\otimes\omega^{\mu'}\otimes\cdots\otimes\omega^{\sigma'}\ \Big|_{\substack{\mu'\mapsto\alpha'\\ \alpha'\mapsto\mu'\\ \sigma'\mapsto\sigma}}\\
&= \underbrace{\left(\partial_\sigma\tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots} + \Gamma^\alpha_{\nu\sigma}\,\tau^{\nu\beta\cdots}_{\alpha'\beta'\cdots} + \cdots - \Gamma^{\mu'}_{\alpha'\sigma}\,\tau^{\alpha\beta\cdots}_{\mu'\beta'\cdots} - \cdots\right)}_{=:\ \nabla_\sigma\tau^{\alpha\beta\cdots}_{\alpha'\beta'\cdots}}\partial_\alpha\otimes\cdots\otimes\omega^{\alpha'}\otimes\cdots\otimes\omega^\sigma
\end{aligned}
$$

The "Riemann tensor"

(R) ∇_μ sat "Leibn rule" in the sense that

$$\nabla_\mu (V^{\alpha\beta\ldots}_{\alpha'\beta'\ldots} W^{\gamma\delta\ldots}_{\gamma'\delta'\ldots}) = (\nabla_\mu V^{\alpha\beta\ldots}_{\alpha'\beta'\ldots}) W^{\cdots}_{\cdots} + V^{\cdots}_{\cdots} \nabla_\mu W^{\cdots}_{\cdots}$$

& similarly if some indices contracted (e.g $\alpha = \gamma'$)
[Follows from fact that ∇ sat Leibniz wrt to $\otimes$ & $\langle\ \rangle$].

(R) Write $\nabla^2 \partial_\mu = R^\alpha_{\mu\beta\gamma}\, \partial_\alpha \otimes \omega^\beta \wedge \omega^\gamma$ $\boxed{w\ R^\alpha_{\mu\beta\gamma} = -R^\alpha_{\mu\gamma\beta}}$ (*) & compute $R^\alpha_{\mu\beta\gamma}$ in terms of Γ

$$\nabla^2 \partial_\mu = \nabla(\Gamma^\nu_{\mu\sigma} \partial_\nu \otimes \omega^\sigma) = \partial_\nu \otimes \omega^\sigma \wedge d\Gamma^\nu_{\mu\sigma} + \Gamma^\nu_{\mu\sigma} \cdot \nabla\partial_\nu \wedge \omega^\sigma + \Gamma^\nu_{\mu\sigma} \partial_\nu \otimes \nabla \omega^\sigma$$

$$= \partial_\nu \otimes \omega^\sigma \wedge \partial_j \Gamma^\nu_{\mu\sigma} \cdot \omega^j + \Gamma^k_{\mu\sigma} \Gamma^\nu_{kj} \partial_\nu \otimes \omega^\sigma \wedge \omega^j - \Gamma^\nu_{\mu k} \Gamma^k_{\sigma j} \partial_\nu \otimes \omega^\sigma \wedge \omega^j$$

$$= (\partial_j \Gamma^\nu_{\mu\sigma} + \Gamma^k_{\mu\sigma} \Gamma^\nu_{kj} - \Gamma^\nu_{\mu k} \Gamma^k_{\sigma j})\, \partial_\nu \otimes \omega^\sigma \wedge \omega^j$$

So $F^\nu_\mu = \partial_j \Gamma^\nu_{\mu\sigma} + \Gamma^k_{\mu\sigma}\Gamma^\nu_{kj} - \Gamma^\nu_{\mu k}\Gamma^k_{\sigma j})\, \omega^\sigma \wedge \omega^j \stackrel{\Delta}{=}$ $\boxed{\text{if } \Gamma^k_{\sigma j} = \Gamma^k_{j\sigma}}$

$\parallel$

$$\sum_{j<\sigma} 2R^\nu_{\mu j\sigma}\, \omega^\sigma \wedge \omega^j = \sum_{j<\sigma} \underbrace{(\partial_j \Gamma^\nu_{\mu\sigma} - \partial_\sigma \Gamma^\nu_{\mu j} + \Gamma^k_{\mu\sigma}\Gamma^\nu_{kj} - \Gamma^k_{\mu j}\Gamma^\nu_{k\sigma})}_{\parallel\ 2R^\nu_{\mu j\sigma}}\, \omega^\sigma \wedge \omega^j$$

(D) If this is the case say ∇ is symmetric

OR: use (*) So $\longrightarrow$

(There's no "2" in books. Not clear why.)

(P) If g is a metric $\exists!$ symmetric connection ∇ s.t. $\nabla g = 0$.
(call ∇ the Levi-Civita conn.)

Pf Uniqueness. $\nabla g = 0 \Rightarrow 0 = \nabla_\sigma g_{\mu\nu} \stackrel{(*)}{=} \partial_\sigma g_{\mu\nu} - \Gamma^i_{\mu\sigma} g_{i\nu} - \Gamma^i_{\sigma\nu} g_{\mu i} \Rightarrow \partial_\sigma g_{\mu\nu} = \Gamma^i_{\mu\sigma} g_{i\nu} + \Gamma^i_{\sigma\nu} g_{\mu i}$

$$\Rightarrow \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu} = \cancel{\Gamma^i_{\nu\sigma} g_{\mu i}} + \Gamma^i_{\nu\mu} g_{\sigma i} + \cancel{\Gamma^i_{\mu\sigma} g_{\nu i}} + \Gamma^i_{\mu\nu} g_{\sigma i} - \cancel{\Gamma^i_{\mu\sigma} g_{\nu i}} - \cancel{\Gamma^i_{\sigma\nu} g_{\mu i}}$$

$$= 2\, g_{\sigma i} \Gamma^i_{\mu\nu}$$

$$\Rightarrow \frac{1}{2} g^{\tau\sigma} \{\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\} = g^{\tau\sigma} g_{\sigma i} \Gamma^i_{\mu\nu} = \delta^\tau_i \Gamma^i_{\mu\nu} = \Gamma^\tau_{\mu\nu}$$

if g^{ij} inv of g_{ij}

Existence: define $\Gamma^\tau_{\mu\nu}$ by & check

(R) So the $n \times n$ matrix of 2-forms $\overbrace{(R^\alpha_{\mu\beta\gamma}\, \omega^\beta \wedge \omega^\gamma)_{\alpha\mu}}^{F^\alpha_\mu}$ is the curvature of ∇

(check if this is the right contraction!!)

D) Ricci tensor $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$; Ricci curvature $R = g^{\mu\nu} R_{\mu\nu} \in \mathcal{O}(M)$

$R_{\mu\nu}\, \omega^\mu \otimes \omega^\nu$

bicanonical form $\eta_g = (\det g_{\mu\nu})(\omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^d)^{\otimes 2} \in (\Omega^d)^{\otimes 2} = K^{\otimes 2}$, $d = \dim M$

(correctly def b/c if $g_{\mu\nu}\, \omega^\mu \otimes \omega^\nu = \tilde{g}_{\tilde\mu\tilde\nu}\, \tilde\omega^{\tilde\mu} \otimes \tilde\omega^{\tilde\nu}$, $\tilde\omega^{\tilde\mu} = \gamma^{\tilde\mu}_{\mu}\, \omega^\mu$

$\Rightarrow$ RHS $= \tilde{g}_{\tilde\mu\tilde\nu}\, \gamma^{\tilde\mu}_{\mu}\, \gamma^{\tilde\nu}_{\nu}\, \omega^\mu \otimes \omega^\nu \Rightarrow g = \gamma \tilde{g} \gamma^T \Rightarrow \det g = (\det \tilde{g})(\det \gamma)^2$;

& $\tilde\omega^1 \wedge \dots \wedge \tilde\omega^d = \det\gamma \cdot \omega^1 \wedge \dots \wedge \omega^d$.

a canonical form is a form $\vartheta \in \Omega^d = K$ s.t. $\vartheta^{\otimes 2} = \pm \eta_g$ (it may not $\exists$ in alg setting ; it $\exists$ on double cover ; it always $\exists$ in C^∞ case $\underbrace{}_{\text{\& is "can"}}$ b/c $\det g_{\mu\nu}$ has const sign & $|\det g_{\mu\nu}|^{1/2}$ is C^∞)

Einstein-Hilbert Lagr. $L_{HE} = R \cdot \vartheta \in \Omega^d(M)$

Hodge star (in C^∞ case) $* = *_g$

$$*\left(\frac{1}{p!}\, \varphi_{i_1 \dots i_p}\, \omega^{i_1} \wedge \dots \wedge \omega^{i_p}\right) = \frac{1}{(d-p)!\, p!}\, \varepsilon_{i_1 \dots i_d}\, |\det g_{\mu\nu}|^{1/2}\, \varphi_{j_1 \dots j_p}\, g^{i_1 j_1} \dots g^{i_p j_p}\, \omega^{i_{p+1}} \wedge \dots \wedge \omega^{i_d}$$

(Note Yang-Mills for $* = *_g$ becomes $L_{YM}(\Phi, A, g)$ while $L_{HE} = L_{HE}(g)$)
(check again!)

Gauge interpr of Einst. theory

$$\begin{tikzcd}
\left\{\begin{matrix}\text{metrics}\\ \text{on } M\end{matrix}\right\} \arrow[r, "="] & \dfrac{GL_d(\mathcal{O}(M))^{\text{symm}}}{g \sim \gamma g \gamma^T} \arrow[l, "\text{if } T_M \text{ triv}"'] \arrow[r] \arrow[d, "\text{in } C^\infty \text{ case}"'] & \dfrac{\Omega^1(M) \otimes \mathfrak{gl}_d(\mathbb{R})}{\text{gauge action of } GL_d^+(M)} \arrow[r, "="] & \left\{\begin{matrix}\text{connections}\\ \text{on } TM\end{matrix}\right\} \arrow[r] & \mathcal{O}(M) \\
& \Omega^d(M) & \text{class of } g \arrow[r, mapsto] \arrow[d, mapsto] & \text{class of the } d \times d \text{ matrix of 1-forms } \left(\Gamma^\nu_{\mu\sigma}\, \omega^\sigma\right)_{\mu\nu} \arrow[r, mapsto] & R \\
& & \vartheta & &
\end{tikzcd}$$

$\Omega^d(M) \ni \vartheta$