GROUPS & QUANTUM MECHANICS

Given $M = \mathbb{R}^n$ (space or space time) $\quad \mathcal{H}$ (complex L^2 fcns)

G Lie group acting on M (hence on $L^2(M)$) and on a fin dim $\mathbb{C}$ vect sp V

$\mathcal{L} : \mathcal{H} \dashrightarrow \mathcal{H}$ a G-equiv linear oper (densely defined !), $\mathcal{H} = \mathrm{Hom}(V, L^2(M))$ G-mod in obv way

Examples 1) $M = \mathbb{R}^3$, $G = SO(3)$, $V = \mathbb{C}$, $\mathcal{L} = \underbrace{-\frac{\hbar^2}{2\mu}\Delta + V}_{\text{Schrödinger}}$, $V = V(r)$

2) $M = \mathbb{R}^4$, $G =$ Lorenz gr, $V = \mathbb{C}^4$ w/ G acting via $\begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$ where $A \in SL(2,\mathbb{C}) \twoheadrightarrow G$

$\mathcal{L} =$ Dirac op.

Back to general sit. Let $\mathcal{R} =$ (clos of) $\mathrm{Ker}\,\mathcal{L}$. Then G acts on $\mathcal{R}$

Define a Casimir operator as an elt of the center of $U(\mathfrak{g})$ where $\mathfrak{g} = \mathrm{Lie}\, G$.

If W is an irred (fin dim) rep of G then any Casimir oper C induces a G-endo of W so (+) is a scalar $q_C\, \mathrm{Id}_W$, $q_C = q_C(W) \in \mathbb{C}$; call q_C the quantum # of W corr to C (*)

Define an eltary particle as an irr rep of G

Often $\mathcal{R} = \bigoplus W_j$ (w/ multipl.), W_j irr. One interprets this by saying that $\forall$ sol of $\mathcal{L}\psi = 0$ is a superpos of eltary particles each of which has some definite quantum #'s.

Say W_i, W_j are irred rep's & $W_i \otimes W_j \simeq \bigoplus V_k$, V_k irred's. One interprets this by saying that $\forall$ state of a pair of particles W_i & W_j is a superpos of particles V_k.

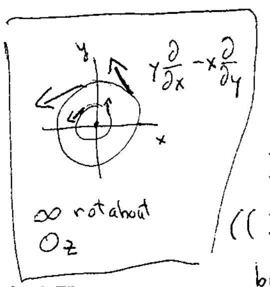
$S^n(W_i)$ represents n bosons of type W_i

$\bigwedge^n W_j$ —"— n Fermions of type W_j

Example $G = SO(3) = \{ A \in GL(3,\mathbb{R}) \mid \det A = 1,\ AA^T = I \}$

$\mathfrak{g} = \{ M \mid \det(I + \varepsilon M) = 1,\ (I+\varepsilon M)(I + \varepsilon M^T) = I \}$

$= \{ M \mid \mathrm{tr}\, M = 0,\ M + M^T = 0 \} = \mathrm{Span} \left\{ \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\gamma_z}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{\gamma_x}, \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{\gamma_y} \right\}$



$I + \varepsilon\gamma_z$ acts on a C^∞ fcn $f(x,y,z)$, $f : \mathbb{R}^3 \to \mathbb{C}$ as follows

$((I + \varepsilon\gamma_z) f)(x,y,z) = f(x + \varepsilon y, -\varepsilon x + y, z) = f(x,y,z) + \varepsilon\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)$

b/c $(I + \varepsilon\gamma_z)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon & 0 \\ -\varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \varepsilon y \\ -\varepsilon x + y \\ z \end{pmatrix}$ So γ_z acts on $C^\infty(\mathbb{R}^3, \mathbb{C})$ as $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = L_z$

& sim for γ_x, γ_y

(+) by Shur's Lemma

(*) Sometimes it is true that if W_1, W_2 have the same quantum #'s (i.e $\forall C$, $q_C(W_1) = q_C(W_2)$) then $W_1 \simeq W_2$ as G-mod's

A Casimir oper (ess the only one) is $C = \gamma_x^2 + \gamma_y^2 + \gamma_z^2 \in U(g)$ [direct verif that it commutes w/ $\gamma_x, \gamma_y, \gamma_z$ using comm rel's b/t $\gamma_x, \gamma_y, \gamma_z$: $\gamma_x\gamma_y - \gamma_y\gamma_x = \gamma_z$ etc.]

Action of C on $C^\infty(\mathbb{R}^3, \mathbb{C})$ is by $\underbrace{L_x^2 + L_y^2 + L_z^2}$ =: "L^2"

In quantum mechanics this op ↗ is called "negative of the total angular momentum" (*)

$$\left(L^2 = 2\left(x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial z^2}\right) + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - xy\frac{\partial^2}{\partial x\partial y} - yz\frac{\partial^2}{\partial y\partial z} - zx\frac{\partial^2}{\partial x\partial z} \right)$$

(L) Let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ & $P_n = \{$ hom pol's in x, y, z of deg n w/ $\mathbb{C}$-coeff $\}$. Then

1) Δ is $SO(3)$-equiv so $SO(3)$ acts on $\mathrm{Ker}(\Delta : P_n \to P_{n-2}) =: V_n$

2) V_n is an irr repr of $SO(3)$ and $\forall$ other irr rep of $SO(3)$ is $\simeq$ to some V_n

Pf 1) clear 2) needs theory.

(P) The quantum number $q_{L^2}(V_n) = -n(n+1)$

Pf $\varphi = (\overset{w}{x+iy})^n \in V_n$ b/c $\Delta\varphi = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\varphi = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\overbrace{n}(x+iy)^{n-1}\overbrace{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x+iy)}^{0}$

Now $\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)w^n = nw^{n-1}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)(x+iy) = nw^{n-1}\underbrace{(y - ix)}_{-i(x+iy)} = -inw^n$

so $\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)^2 w^n = -n^2 w^n$. Similarly one computes $\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)^2 w^n = \cdots$ etc

& one gets. $L^2 w^n = -n(n+1)w^n$.

(R) V_n appears ∞ many times in $\underset{}{L^2(\mathbb{R}^3)} = L^2(\mathbb{R}^3, \mathbb{C})$. Indeed $\forall$ fcn $f \in L^2(\mathbb{R}_+)$, $f(\overbrace{x^2+y^2+z^2}^{r^2}) \cdot V_n \subset L^2(\mathbb{R}^3)$, $\cong V_n$

(L) For $\varphi \in C^\infty(\mathbb{R}_+)$, $g \in V_n$, $\Delta[\varphi(r) \cdot g] = (L_n\varphi)(r) \cdot g$ where $L_n\varphi = \varphi'' + \frac{2(n+1)}{r}\varphi' - n(n+1)\varphi$.

Pf. Direct comp (hope didn't mk a mistake) (Use $\Delta g = 0$, $\mathcal{E}g = ng$, $\mathcal{E} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$)

(D) Let $g_{nj} \in V_n$ be a basis of V_n, $j = 1, 2, \ldots, 2n+1$

(L') $\left\{ \sum_{nj} \varphi_{nj}(r) \cdot g_{nj} \mid \varphi_{nj} \in C^\infty(\mathbb{R}_+) \right\}$ is dense in $L^2(\mathbb{R}^3)$.

[Pf. Use Weierstr-Stone plus easily checked fact that $P_n = V_n \oplus r^2 V_{n-2} \oplus r^4 V_{n-4} \oplus \ldots$]

(R) Using the above one can study spectrum of $\mathcal{L} = -\frac{\hbar^2}{2\mu}\Delta + V$, $V = V(r)$ as follows

. Want to study eq $\mathcal{L}\psi = E\psi$, $E \in \mathbb{R}$. Assume $\psi = \sum \varphi_{nj}(r) \cdot g_{nj}$.

Eqn becomes $-\frac{\hbar^2}{2\mu}\sum (L_n\varphi_{nj})(r) g_{nj} + \sum V(r)\varphi_{nj}(r) g_{nj} = E\sum \varphi_{nj}(r) g_{nj}$

Enough to solve $\left(-\frac{\hbar^2}{2\mu}L_n + V(r) - E\right)\varphi_{nj} = 0$ etc?

For $V = \frac{ct}{r^2}$ this leads to Bohr's atom spectrum. (see Sternberg's book for details)

Strategy: for each n find spectrum $\sigma\left(-\frac{\hbar^2}{2\mu}L_n + V\right) = \{E_{n1}, E_{n2}, \ldots\}$

Any elt $E = E_{nm}$ in the union of these spectra is in $\sigma(\mathcal{L})$ b/c if $\left(-\frac{\hbar^2}{2\mu}L_n + V\right)\varphi = E\varphi$ then $\mathcal{L}(\varphi \cdot g_n) = E_{nm}\varphi \cdot g$

Conversely one proves using (L') that $\forall$ elt in $\sigma(\mathcal{L})$ is the union of $\sigma\left(-\frac{\hbar^2}{2\mu}L_n + V\right)$

(*) Indeed if p_x, p_y, p_z are classical momenta then classical angular momenta are $yp_x - xp_y, \ldots$

In quant mech, on the other hand, p_x is repl by $i\frac{\partial}{\partial x}$, etc. So the class total ang mom $(yp_x - xp_y)^2 + \cdots$ is repl by $-L^2$.