1

History of p-adic analysis

(Sign means p-adic interpol)

Prehistory

17C Bernoulli (Jacob) defined $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$ (Ber #'s) & computed $1^k+2^k+\dots+n^k$ in terms of $B_1,\dots,B_k$ & pved a # of id's (as we shall see)

18C Euler "proved" $\zeta(1-k) := \sum_{n=1}^{\infty} n^{k-1} = -\frac{B_k}{k}$, $k \geq 2$. $\Big[\sum_{k=0}^{\infty} \zeta(-k)\frac{t^k}{k!} = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} n^k \frac{t^k}{k!} = \sum_{n=1}^{\infty} e^{nt}$ (everything here diverges) $= \frac{e^t}{1-e^t} = -1 - \frac{1}{e^t-1}\Big]$

19C Riemann; $\zeta(s) = \sum n^{-s}$ mero on $\mathbb{C}$ & mks sense & correct.
(Δ) pved $\Lambda(1-s) = \Lambda(s)$, $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ (*)

19C Kummer defined $Cl(\mathbb{Q}(\zeta_N)) = \frac{\{\text{fr ideals of } \mathbb{Q}[\zeta_N]\}}{\{\text{pr ideals of } \mathfrak{O}\}}$, $h_N = \# Cl(\mathbb{Q}(\zeta_N))$

defined p regular if $p \nmid h_p$; irreg otherwise.

pved $p \mid h_p \iff p \mid$ numerator of $B_2, B_4, B_6, \dots, B_{p-3}$

pved $p \nmid h_p \Rightarrow$ FLT ok for p

conj: $\exists\infty$ regular p's. (known $\exists\infty$ irreg pr's) follows from

pved $m \equiv n \not\equiv 0 \ (p-1) \Rightarrow \frac{B_m}{m} \equiv \frac{B_n}{n} \ (p)$ "Kummer congr's"

History $\in$ 20C

Hensel defined
- $\mathbb{Q}_p$ as completion(☺) of $\mathbb{Q}$ wrt distance $d_p(x,y) = |x-y|_p$, $\left|\frac{a}{b}\right|_p = p^{v_p(a)-\,\dots}$,
- $\mathbb{Z}_p$ as $\{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ subring!

1918 Ostrovski defined norm on a field F as a map $\|\ \| : F \to \mathbb{R}_+$ s.t. $\|x\| = 0$ iff $x = 0$, $\|xy\| = \|x\|\|y\|$, $\|x+y\| \leq \|x\| + \|y\|$

norm def metric $\|x-y\|$, 2 norms are equiv if the 2 metrics def same top.

pved (T) $\forall$ non-triv(††) norm on $\mathbb{Q}$ is equiv to $|\ |$ or $|\ |_p$

So $\mathbb{Q} \hookrightarrow \mathbb{R}, \mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_5, \dots$ } philosophical principle: treat these on = footing

Hasse pvd: A quadr form $\sum a_{ij}x_ix_j$ ($a_{ij} \in \mathbb{Q}$) has a non-triv zero in $\mathbb{Q}^n$ iff it has a non-triv zero in $\mathbb{R}^n$ & $\mathbb{Q}_p^n$ (all p). (Hasse-Mink principle fails for cubics etc)

☺ {Cauchy seq's}/{those $\to 0$}

†† i.e. top not discrete.

Δ idea $\zeta = M\theta$ (Mellin transform)

(*) define!

top-group

Chevalley defined the adeles ring $\mathbb{A}_{\mathbb{Q}} = \{(a_\infty, a_2, a_3, a_5, \dots) \in \mathbb{R} \times \prod_p \mathbb{Q}_p \mid |a_p|_p \le 1 \text{ aap}\}$

(used it & "ideles" to reformulate class field th).

50s Tate's thesis Real Harmonic analysis on gr of adeles to pv fcnal eqns for Riemann ζ & var's.

(measures are real valued : classical meas-th.)

60s Tate Rigid analytic geometry : $\mathbb{C}_p$ & an of $\mathbb{C}$-an geo

1964 Kubota-Leopold pvd $\exists$ of a "p-adic analytic fcn" $\zeta_p(s) : \mathbb{Z}_p \to \mathbb{Z}_p$ s.t. (i.e. "given by a series")

$\zeta_p(1-k) = (1-p^{k-1})\zeta(1-k)$ ["p-adic interpol" of $(1-p^{-s})\zeta(s)$]

reinterpret of Kummer's congr's.

similar for Dir L-fcns $L(s) = \sum \chi(n) n^{-s}$. MIRACLE !!!

define : p-adic interp of a $\mathbb{C}$-an. fcn & conj : all important fcns

$\boxed{\text{P-adic ghosts of complex objects}}$ $\to$ à la ζ (eg L) have p-ad. int.

60's Mazur, Manin – Interpreted Kubota-Leopold ζ_p as a "p-adic Mellin transf" of "p-adic measures" (meas's are $\mathbb{Q}_p$-valued : non-class meas. th.)

found p-ad interp's of L-fcns of ell curves.

60's Iwasawa's idea : view $\varprojlim_{\text{Norm}} Cl(\mathbb{Q}(\zeta_{p^n})) + \mathbb{Z}_p$-action + a certain p-ad an L fcn (2 candidates are equal: by interpo, as char pol of action), as an analogue of

$\mathbb{Z}_p = G(\mathbb{Q}(\zeta_{p^\infty}) | \mathbb{Q}(\zeta_p))$

("magnificent analogy")

$Cl^0(X) = Jac(X)$ curve / fin field + action of Frob + Weil ζ-fcn of X

(2 candidates are equal: by counting pts, by char pol of action)

60's Dwork (triggered by Tate Rig an) p-adic analysis of diff eqns $\frac{dy}{dx} = A(x)y$ ∇

p-adic pf of rationality of Weil's zeta.

90's Wiles etc

Myself $\delta : \mathbb{Z}_p \to \mathbb{Z}_p$ in place of $\frac{d}{dx}$ in Dwork.

$\mathbb{Z}_p$ —"— {p-adic an fcns $\mathbb{Q}_p \to \mathbb{Q}_p$}

etc.

∇ Sol's of such (w/ $A(x) \in M_{n \times n}(\mathbb{Q}(x))$) in p-ad analytic fcns $\mathbb{Q}_p \to \mathbb{Q}_p$.