Interior, exterior, and Lie derivatives

~~Algebraic differential calculus; de Rham & Lie~~

A ring, E an A-module, $\Lambda^r E = \dfrac{E^{\otimes r}}{\langle x_1 \otimes \cdots \otimes x_r \mid x_s = x_t, \text{ some } s \neq t \rangle}$

(T) $\mathrm{Hom}_A(\Lambda^r E, M) \overset{alt}{\cong} \mathrm{Alt}_A(E^r, M)$ for M A-mod.

(R) $\exists$ nat $\Lambda^r E \times \Lambda^s E \to \Lambda^{r+s} E$ induced by $\otimes$.

(R) If $v \in \check{E} = \mathrm{Hom}_A(E, A)$ get $i_v : \Lambda^r E \to \Lambda^{r-1} E$, $(i_v \varphi)(v_1, \dots, v_{r-1}) = \varphi(v, v_1, \dots, v_{r-1})$ — contraction

& $\check{\check{E}} = E$ E free so f.g.

$\varphi \in \Lambda^r E = \Lambda^r \check{\check{E}} = (\Lambda^r \check{E})^\vee \cong \mathrm{Alt}_A(\check{E}^r, A) \ni \check{\varphi}$

(R) Above generalizes to projective f.g E by localizing /Noeth A

(D) Let $R \to A$ be a ring homo, $\Omega_{A/R} = \dfrac{\{\text{free } A\text{-mod gen by } da,\ a \in A\}}{\langle \text{Leibnitz, etc} \rangle}$ $d \leftarrow A$

(T) $\mathrm{Hom}_A(\Omega_{A/R}, M) \cong \mathrm{Der}_R(A, M) := \dots$

(D) $T_{A/R} = \mathrm{Der}_R(A, A) = \check{\Omega}_{A/R}$, $\Omega^i_{A/R} = \Lambda^i \Omega_{A/R}$ — de Rham complex

(T) $\exists!$ R-mod homos d, $A \xrightarrow{d} \Omega \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \to \dots$ s.t. $\forall \omega \in \Omega^i, \eta \in \Omega^j$ have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^i \omega \wedge d\eta$. Also $d \circ d = 0$ (graded der)

(D) Ass. $R = \mathbb{R}$ & $A = C^\infty(\text{dom in } \mathbb{R}^n)$ or R Noeth, $A \overset{ét}{\supset} R[x_1, \dots, x_n]$ & set $\Omega^i = \Lambda^i \check{T}_{A/R}$ (*)

(D) In either case let $v \in T_{A/R}$. Since Ω free, f.g, i_v well def. (**)

Define Lie derivative $L_v : \Omega^i \to \Omega^i$ by $L_v = i_v d + d i_v$ (**)
(H. Cartan formula*)

(EX) 1) $i_v(\omega \wedge \eta) = (i_v \omega) \wedge \eta + (-1)^i \omega \wedge i_v(\eta)$, $\omega \in \Omega^i, \eta \in \Omega^j$ (graded der)

2) $L_v(\omega \wedge \eta) = (L_v \omega) \wedge \eta + \omega \wedge i_v(\eta)$ —"— —"— (der)

3) $L_v d = d L_v$ (clear)

In either case $T_{A/R}$ is free, f.g. gen by $\frac{\partial}{\partial x_i}$ — Helgason p.10. hence $\Omega^1 = \Omega$ is also free (gen by dx_i)

In 2nd case $\Omega^i = \Omega^i_{A/R}$

(*) In 1st case $\Omega^i \neq \Omega^i_{A/R}$ a priori.

(**) Note $i_v(df) = v(f)$ for $f \in A = \Omega^0$ so $L_v f = v(f)$ for $f \in A = \Omega^0$, $L_v(df) = d(v(f))$