Finite generation of invariants

(T) (Artin-Tate) Ass A Noeth ring, S a fin gen A-alg and $T \subset S$ sub A-alg. Assume S fin / T. Then T is fin gen / R

$A \overset{\text{fin gen}}{\subset} T \underset{\text{fin}}{\subset} S \Rightarrow A \underset{\text{fin gen}}{\subset} T$

Pf. Let $S = A[\sigma_1, \dots, \sigma_r]$, $S = \sum_j T s_j$. So $\sigma_i = \sum t_{ij} s_j$; $s_i s_j = \sum t'_{ijk} s_k$.

Let $T_0 = A[\dots, t_{ij}, \dots, t'_{ijk}, \dots] \subset T$.

Claim 1. $S = \sum_j T_0 s_j$ [indeed $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_e} = (\sum_{j_1} t_{i_1 j_1} s_{j_1})\cdots(\ ) \overset{\text{ind}}{=} T_0$-lin comb of s_k's

Claim 2. T fin / T_0 [indeed T_0 Noeth b/c fin gen / A so S Noeth T_0-mod so T fin gen T_0-mod]

Write $T = \sum_j T_0 \tau_j$.

Claim 3. $T = A[\dots, t_{ij}, \dots, t'_{ijk}, \dots, \tau_j, \dots]$ clear.

(C) k field, A fin gen k-alg, $G < \mathrm{Aut}_k A$ fin subgp. Then A^G fin gen / k.

Pf $k \subset A^G \subset A$, $A^G \subset A$ integral b/c $\forall a \in A$ root of $\prod_{\sigma \in G}(x - \sigma a) \in A^G[x]$.

But A fin gen / A^G so A fin / A^G & done by $\Uparrow$.

~~(D) A rep of a gp G is a homo $G \xrightarrow{\rho} GL(V)$, V fin dim k-lin sp. ρ called compl. reducible if $V \cong \oplus V_i$, V_i irr~~ So $G \curvearrowright V$. ~~ρ called irr if no proper $\neq 0$ G-inv subsp.~~

(D) k field, G gp $\curvearrowright A$, A a k-alg, A^G. A Reynolds operator for $G \curvearrowright A$ is a map $R: A \to A^G$ which is id on A^G and an A^G-mod map.

(T) (Hilbert) Let G gp (not nec fin) $\curvearrowright A = k[x_1, \dots, x_N]$. Ass $\exists$ Reynolds op (sending $A_m \to A_m$ (call it homog)). Then A^G fin gen / k

Pf. Note A^G graded (xplu: $A^G = \oplus A_i^G$, $A_i^G \cap A_i =$ hom pol of deg i). Let $A_+^G = \oplus_{i>0} A_i^G$.

Basis th $\Rightarrow$ $A_+^G \cdot A = \sum_{j=1}^m A \cdot f_j$. May ass $f_j \in A_{m_j}^G$. Claim: $A^G = k[f_1, f_2, \dots, f_m]$

Pv by ind on n that $A_n^G \subset k[f_1, \dots, f_m]$. Take $f \in A_n^G$. Then

$f = \sum a_j f_j$. May ass $a_j \in A_{n-m_j}$. Have $f = Rf = \sum \underbrace{(Ra_j)}_{\in A^G_{n-m_j}} f_j$ & apply ind to Ra_j.

(T) (Hilbert) ch $k = 0$ & $SL_n(k) \curvearrowright k[x_1, \dots, x_N]$ linearly (xplu). Then $\exists$ homog Reynolds op.

(C) ch $k = 0$ & $SL_n(k) \curvearrowright k[x_1, \dots, x_N]$ linearly. Then $k[x_1, \dots, x_N]^{SL_n(k)}$ fin gen / k.

Pf of $\mathbb{C}$ Step 1 $SL_n(\mathbb{C})$-inv $\Leftrightarrow$ $SU(n)$-inv [look @ Lie alg's]

Step 2 $R(f) = \frac{1}{V}\int_{SU(n)} f^\sigma d\sigma$, V vol.

} sketch