Jordan normal form etc

(D) R ring, $A, B \in M_n(R)$. $A \approx B \iff \exists\, U, V \in GL_n(R),\ B = UAV$

$A \sim B \iff \exists\, U \in GL_n(R),\ B = UAU^{-1}$

(P) R PID, $A \in M_n(R) \Rightarrow A \approx D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$. [Pf. (Eucl. case): Gauss elim.]

(C) k field $\Rightarrow$ any $A \in M_n(k)$ is $\approx$ to $\begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & 0 \end{bmatrix}$; so $M_n(k)/\approx\ \{0, 1, \dots, n\}$.

(Q) what is $M_n(k)/\sim$? Answer: via Jordan can form.

(D) If $A \in M_n(k)$ then $k^n =: V$ is a $k[x]$-module via $Av =: x \cdot v$, $v \in k^n$ = col. vects. (*) write $V_A \in \mathrm{Mod}_{k[x]}$

(L) For $A, B \in M_n(k)$ we have $A \sim B \iff V_A \simeq V_B$ [Pf. $V_A \simeq V_B \iff \exists\, k^n \xrightarrow{U} k^n$, $v \mapsto Uv$ s.t. $x \cdot Uv = U x v$ i.e. $AU = UB$]

(T) Let $A \in M_n(k)$, $xI - A \in M_n(k[x])$, $xI - A \approx D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ in $M_n(k[x])$. Then $V_A \simeq \bigoplus_{i=1}^{n} \frac{k[x]}{(d_i)}$ as $k[x]$-mod's. (C) if $xI - B \approx \begin{bmatrix} e_1 & & \\ & \ddots & \\ & & e_n \end{bmatrix}$ then $A \sim B \iff \bigoplus \frac{k[x]}{(d_i)} \overset{\odot}{\simeq} \bigoplus \frac{k[x]}{(e_i)}$

Pf. Write $xI - A = UDV$. Get (**)

$$\begin{tikzcd}
k[x]^n \arrow[r, "{[xI-A]}"] & k[x]^n \arrow[r, "\pi\ (\odot)"] & k^n & \text{(semiexact ♡)} \\
k[x]^n \arrow[u, "{[V^{-1}]}"] \arrow[r, "{[D]}"'] & k[x]^n \arrow[u, "{[U]}"] \arrow[r] & \operatorname{coker}[D] \arrow[u, dashed, "\varphi"'] & \text{(exact)}
\end{tikzcd}$$

(φ exists b/c $\pi \circ [U]$ is 0 on $\operatorname{Im}[D]$). Clearly π surj. So φ surj. I'm done if I show $\dim \operatorname{coker}[D] = n$ (b/c $\operatorname{coker}[D] = \bigoplus \frac{k[x]}{(d_i)}$). But has dim $=$ $\sum \deg d_i = \deg \prod d_i = \deg(\det D) = \deg(\det A) = n$. □

(T) Say $k = \bar{k}$. Then $\forall A \in M_n(k)$ is $\sim \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$, $J_i = \begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ 0 & & \lambda \end{bmatrix}$ "Jordan block".

Pf. Decompose $\frac{k[x]}{d_i}$ into $\oplus$ of $\frac{k[x]}{(x-\lambda)^m}$ & take basis $1, x-\lambda, (x-\lambda)^2, \dots$ □

$\odot$ $\pi e_i = e_i$ i.e. $\pi(\sum f e_i) = \sum f_i \cdot e_i = \sum f_i(A) e_i$

♡ $\pi([xI - A] e_1) = \pi\left(\begin{bmatrix} x - a_{11} \\ -a_{21} \\ \vdots \\ -a_{n1} \end{bmatrix}\right) = \pi((x - a_{11}) e_1 - a_{21} e_2 - \cdots)$
$= (x - a_{11}) \cdot e_1 - a_{21} e_2 - \cdots = 0$

(**) $\forall C \in M_n(R)$ defines $R^n \xrightarrow{[C]} R^n$ ("col vects"), $v \mapsto Cv$. Note $[D] \circ [C] = [DC]$

$\odot$ can be decided as foll: decompose $d_i = \prod p^{n_i(p)}$ etc.

(*) so if $e_1, \dots, e_n$ stand basis then $x \cdot e_i = A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} = a_{1i} e_1 + \cdots + a_{ni} e_n = \sum_j a_{ji} e_j$

$/\!/ \ \Phi_A(x) = x^n - (\mathrm{Tr}A)x^{n-1} + \cdots + (-1)^n \det A$ ⊙

(D) Char polynomial (+): $\det(xI - A) \in k[x]$

Minimal polyn (+) $\psi_A \in k[x]$ s.t. $(\psi_A) = \{f \in k[x] \mid f(A) = 0\} = \{f \in k[x] \mid f \cdot V_A = 0\}$

(P) (in not above) $\Phi_A = \prod_{i=1}^{n} d_i$, $\psi_A = d_n$ (provided $d_1 | d_2 | \cdots | d_n$).

(C) $\psi_A \mid \Phi_A$

(C) (Hamilton-Cayley) $\Phi_A(A) = 0$

(D) A diagonalizable (or semisimple) if $A \sim$ diagonal mat in $M_n(k)$.

$k=\bar{k}$ (P) A diagonalizable $\iff \psi_A$ has simple roots.

Pf. A diag $\Rightarrow V_A \simeq \bigoplus \frac{k[x]}{x-\lambda_i} \Rightarrow \psi_A = \prod$ of the distinct $x - \lambda_i$

ψ_A simple roots $\Rightarrow d_i$ have simple roots $\xRightarrow[\text{Chin remain}]{} V_A \simeq \bigoplus \frac{k[x]}{x-\lambda} \Rightarrow A \sim$ diagonal.

$k=\bar{k}$ (C) Say $\underset{k^n}{\underset{\shortparallel}{W \subset V}}$ subspace, $\underset{M_n(k)}{\underset{\cap}{A}}$-invariant. Say A diagonalizable. Then $A|_W$ diagonalizable.

Pf. $\psi_A(A) = 0 \Rightarrow \psi_A(A|_W) = 0 \Rightarrow \psi_{A|_W} \mid \psi_A \Rightarrow \psi_{A|_W}$ has simple roots.

$k=\bar{k}$ (T) Assume $A, B \in M_n(k)$ both diagonalizable & $AB = BA$. Then A & B simultaneously diagonalizable.

Pf. Let λ eigv of A, $V_\lambda = \{v \in V \mid Av = \lambda v\}$. Then $BV_\lambda \subset V_\lambda$ (b/c if $v \in V_\lambda \Rightarrow Av = \lambda v \Rightarrow ABv = BAv = B\lambda v = \lambda Bv \Rightarrow Bv \in V_\lambda$). By ↑ $B|_{V_\lambda}$ diagonalizable so $\exists$ basis of V_λ wrt which B diag (& also A diag b/c scalar) Now $V = \bigoplus$ of V_λ's. □

⊙ roots are called eigenvalues. Equiv λ eigv if $\exists\, v \neq 0$ s.t. $Av = \lambda v$.

(+) invariants under $\sim$