

# Langlands philosophy

Ⓓ Dirichlet L-func  $\sum_{n \geq 1} \frac{a_n}{n^s} = L(s)$ ,  $a_n \in \mathbb{C}$

ⒺⒺ  $\zeta(s) = \sum \frac{1}{n^s}$

Ⓓ Euler prods  $L(s) = \prod_p \Phi_p(p^{-s})^{-1}$ ,  $\Phi_p \in \mathbb{C}[X]$

ⒺⒺ  $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$

Ⓓ 4 classes of L-func

1) Motivic L-func  
(from geometry)

$L(X, s)$

$A = \mathbb{Z}[X_1, \dots, X_n] / (F_1, \dots, F_m)$ ,  $X = \text{Spec } A$ ,  $L(X, s) = \prod_{P \in \text{Max } A} (1 - N(P)^{-s})^{-1}$   
 where  $N(P) = \#(A/P)$ . Due to Riem/Eul  $A = \mathbb{Z}$ , Dedekind  $A = \mathcal{O}_K$ , Artin  $A = \#_p [x, y]/(f)$ , Hasse-Weil (gen)  
 Artin-Weil-Deligne  $\Rightarrow L_p(X, s) = \prod_p R_p(p^{-s})$ ,  $R_p \in \mathbb{C}(X)$   
 (Fields) (sort of Euler prod) + info on zero & poles of  $R_p$  ("Riem hyp")

$L(A, s) = \zeta_A(s)$

$L(X, s) = \prod_{P \in \text{Max } A} (1 - N(P)^{-s})^{-1}$

see next page

2) Motivic L-func  
(from arith.)

$L(\rho, s)$

$\rho: G(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_n(L)$ ,  $L = \mathbb{C}, \mathbb{Q}_\ell$ , Gal rep (cont)  
 For  $L = \mathbb{C}$ , Artin:  $L(\rho, s) = \prod_p \det(I - \rho(\sigma_p) p^{-s})^{-1}$

3) Adic L-func (adelic)

$L(X, s)$

For  $L = \mathbb{Q}_\ell$  similar under extra cond. (rationality & unramified fin many prs)  
 Frob elt

Kron-Hecke

4) Automorphic L-func

(modular)

$L(f, s)$

$f$  modular form i.e.  $f: \mathbb{H} \rightarrow \mathbb{C}$  s.t. for  $\Gamma < SL_2(\mathbb{Z})$   
 1)  $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ ,  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  (fin ind)  
 2)  $f$  "holo @ cusps" ( $\Rightarrow f(z) = \sum_{n \geq 0} c_n q^n$ ,  $q = e^{2\pi i z/h}$ )  
 For  $f$  cusp form if  $c_0 = 0$ .  
 Some int

Hecke:  $L(f, s) = (\frac{2\pi}{h})^{-s} \pi(s) \sum_{n \geq 1} \frac{c_n}{n^s} = m(f(iy))$

Mellin tr of  $u: \mathbb{R}_+ \rightarrow \mathbb{C}$   
 $(Mu)(s) = \int_0^\infty x^s u(x) \frac{dx}{x}$   
 $\Rightarrow L(f, s)$  has a functional eq

if  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma \Rightarrow f(z+1) = f(z) \Rightarrow f = \sum_{n \in \mathbb{Z}} c_n q^n$ ,  $q = e^{2\pi i z}$

This is an Euler product

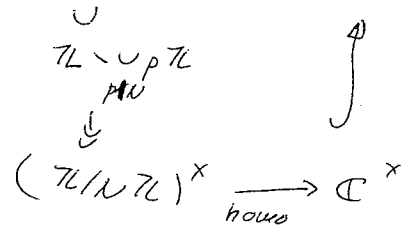
$= \prod_p L_p(X, s)$ ,  $L_p(X, s) = L(X \otimes_{\mathbb{F}_p} s) = Z(p^{-s})$ ,  $Z(t) = \exp \left[ \sum_{\ell=1}^{\infty} N_\ell \frac{t^\ell}{\ell} \right]$

$N_\ell = \#X(\mathbb{F}_{p^\ell})$ . If  $N_\ell = \text{Tr}(A^\ell) \Rightarrow Z(t) = \det(I - tA)^{-1}$ . So need coh th giving  $A$ .

f: ex of aut fnc. replacing  $GL_2$  by other grps like  $GL_n$  & f by "automorphic repn  $f = \prod$ "

3) Art L-funs (adelic)

Example = Dir class  $\chi : \pi \longrightarrow \mathbb{C}$

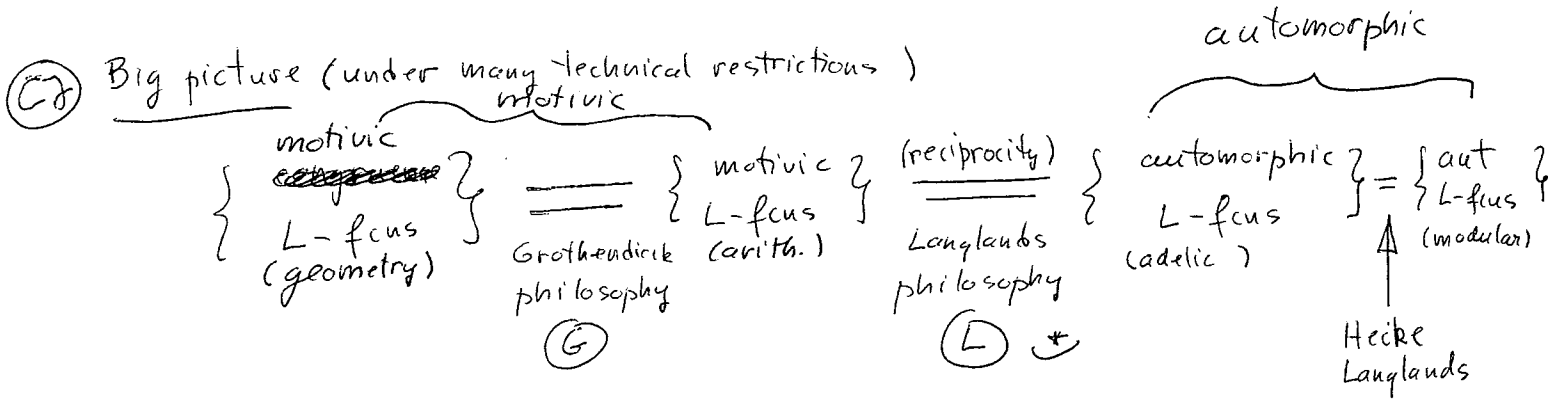


$$\boxed{L(\chi, s)} = \sum \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1} \quad (\text{Dir L-funs})$$

Can be generalized to  $\chi : \text{div classes of } \mathbb{F}\text{-fields} \rightarrow \mathbb{C}$

$\chi : \text{adelic grs} \rightarrow \mathbb{C}$

$\mathbb{A}/\mathbb{Q}^{\times}$  etc.



EG1 If  $A = \mathbb{Z}$  have  $L(A, s) = L(f, s) = \zeta(s) = L(\chi, s)$

$f = \text{triv}$   
 $f = \theta(z) = \sum_{n \geq 1} q^n, q = e$   
 $\chi = 1$

$\uparrow$  tautol.  
 $\uparrow$  Euler prod for  $L(1, s)$   
 $\uparrow$  Riemann's big discovery

EG2 (Quadratic reciprocity in a simple sit.)

$A = \mathbb{Z}[x, y] / (x^2 + y^2 - 1)$  have  $L(A, s) = L(f, s) = L(\chi, s) = L(f, s)$

$f: G(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow G(\mathbb{Q}(i)/\mathbb{Q}) \hookrightarrow \mathbb{C}^\times$   
 $\uparrow$  triv comp  
 $\uparrow$  Hecke  
 $\uparrow$  Euler's prod + quadr rec

$(\frac{-1}{p}) = (-1)^{\frac{p-1}{2}} = \chi(p)$   
 $\chi: \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow (\frac{\mathbb{Z}}{4\mathbb{Z}})^\times \cong \{1, 3\} \cong \{1, -1\}$   
 Dir Mar

$1 - \frac{1}{3^s} + \frac{1}{5^s} - \dots$   
 $\uparrow$  a sort of  $\theta$  for  $\mathbb{Z}$

(R) [ ] is a "reciprocity step"

EG3 Quadr rec & more generally "Artin rec"

EG4  $\mathbb{G}$  "C" given by Hasse-Tate mods  $\lim \leftarrow E[e^n]$  for  $A = \frac{\mathbb{Z}[x, y]}{(y^2 - \varphi_3(x))}$

or e't coh for arb A

$\mathbb{G}$  "C" needs hypotheses idea: varieties can be broken up in coh. pieces.

EG5  $\mathbb{L}$  "C" Eichl-Sh  
 $\mathbb{L}$  "C" Wiles-Taylor ) for

\* in partic motivic L-fcus have fcnal equs (bk auto L-fcus do)