$\boxed{\text{Langlands philosophy}}$

(D) Dirichlet L-fcns $\sum_{n\geq 1} \frac{a_n}{n^s} = L(s)$, $a_n \in \mathbb{C}$ (EG) $\zeta(s) = \sum \frac{1}{n^s}$

(D) Euler prods $L(s) = \prod_p \Phi_p(p^{-s})^{-1}$, $\Phi_p \in \mathbb{C}[x]$ (EG) $\zeta(s) = \prod_p (1-p^{-s})^{-1}$

(D) 4 classes of L-fcns

$\boxed{L(A,s)} = \boxed{\zeta_A(s)}$

1) Motivic L-fcns (from geometry) $\boxed{L(X,s)}$

$A = \frac{\mathbb{Z}[x_1,\dots,x_n]}{(F_1,\dots,F_m)}$, $X = \operatorname{Spec} A$, $\boxed{L(X,s)} = \prod_{P\in \operatorname{Max} A} (1-N(P)^{-s})^{-1}$ (*)

where $N(P) = \#(A/P)$. Due to Riem/Eul $A=\mathbb{Z}$, Dedekind $A = \mathcal{O}_K$, Artin $A = \mathbb{F}_p[x,y]/(f)$, Hasse-Weil (gen)

Artin-Weil-Deligne (Fields) $\Rightarrow$ $L_p(X,s) = R_p(p^{-s})$ (sort of Eul prod), $R_p \in \mathbb{C}(x)$ + info on zer & poles of R_p ("Riem hyp")

2) Motivic L-fcns (from arith.) $\boxed{L(\rho,s)}$

3) Aut L-fcns (adelic) $\boxed{L(\chi,s)}$

see next page

$\rho: G(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(L)$, $L = \mathbb{C}, \mathbb{Q}_\ell$, Gal rep (cont)

For $L = \mathbb{C}$, Artin: $\boxed{L(\rho,s)} \stackrel{\cdot}{=} \prod_p \det(I - \rho(\sigma_p)p^{-s})^{-1}$ (**)

Frob elt

For $L = \mathbb{Q}_\ell$ similar under xtra cond (rationality & unram outside fin many prs)

4) Automorphic L-fcns (modular) $\boxed{L(f,s)}$

Kron-Hecke

f modular form i.e. $f: \mathbb{H} \xrightarrow{hol} \mathbb{C}$ s.t. for $\Gamma < SL_2(\mathbb{Z})$ (fin ind)

1) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$, $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma$

2) f "holo @ cusps" ($\stackrel{***}{\Rightarrow}$ $f(z) = \sum_{n\geq 0} c_n q^n$, $q = e^{2\pi i z/h}$, h some int)

For f cusp form if $c_0 = 0$.

Hecke; $\boxed{L(f,s)} = \left(\frac{2\pi}{h}\right)^{-s}\Gamma(s)\sum_{n\geq 1}\frac{c_n}{n^s} = \mathcal{M}(f(iy))$

Mellin tr of $u: \mathbb{R}_+ \to \mathbb{C}$
$(\mathcal{M}u)(s) = \int_0^\infty x^s u(x) \frac{dx}{x}$
$\Rightarrow L(f,s)$ has a fcnal eq

f: ex of aut fcn. replacing GL_2 by other grs G like GL_n & f by "automorphic repn $f = \pi$" of G.

(***) if $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} \in \Gamma \Rightarrow f(z+1) = f(z) \Rightarrow f = \sum_{n=-\infty}^{\infty} c_n q^n$, $q = e^{2\pi i z}$.

(**) This $\underline{\underline{\text{is}}}$ an Euler product

(*) $= \prod_p L_p(X,s)$, $L_p(X,s) = L(X\otimes \mathbb{F}_p, s) \stackrel{\text{easy}}{=} Z(p^{-s})$, $Z(t) = \exp\left[\sum_{\ell=1}^\infty N_\ell \frac{t^\ell}{\ell}\right]$

$N_\ell = \#X(\mathbb{F}_{p^\ell})$. If $N_\ell = \operatorname{Tr}(A^\ell) \Rightarrow Z(t) = \det(I - tA)^{-1}$. So need coh th giving A.

3) <u>Aut L-fcns (adelic)</u>

Example : Dir chars χ :

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \mathbb{C} \\ \mathbb{Z}\setminus\bigcup_{p\mid N} p\mathbb{Z} \arrow[u, phantom, "\cup"] \arrow[d, twoheadrightarrow] & \\ (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\text{homo}"'] & \mathbb{C}^\times \arrow[uu, hook] \end{tikzcd}$$

$$\boxed{L(\chi,s)} = \sum{}' \frac{\chi(n)}{n^s} = \prod_p (1-\chi(p)p^{-s})^{-1} \qquad \text{(Dir L-fcns)}$$

Can be generalized to χ : div class grs of #-fields $\to \mathbb{C}$

χ : $\underbrace{\text{adelic}}_{\mathbb{A}^\times/\mathbb{Q}^\times \text{ etc.}}$ grs $\to \mathbb{C}$

(C) Big picture (under many technical restrictions)

$$\overbrace{\left\{\begin{matrix}\text{motivic}\\ \text{L-fcns}\\ \text{(geometry)}\end{matrix}\right\} \underset{\substack{\text{Grothendieck}\\ \text{philosophy}\\ (G)}}{=\!=} \left\{\begin{matrix}\text{motivic}\\ \text{L-fcns}\\ \text{(arith.)}\end{matrix}\right\}}^{\text{motivic}} \underset{\substack{\text{Langlands}\\ \text{philosophy}\\ (L)\ (*)}}{\overset{\text{(reciprocity)}}{=\!=}} \overbrace{\left\{\begin{matrix}\text{automorphic}\\ \text{L-fcns}\\ \text{(adelic)}\end{matrix}\right\} \underset{\substack{\uparrow\\ \text{Hecke}\\ \text{Langlands}}}{=} \left\{\begin{matrix}\text{aut}\\ \text{L-fcns}\\ \text{(modular)}\end{matrix}\right\}}^{\text{automorphic}}$$

(EG1) If $A = \mathbb{Z}$ have $L(A,s) \underset{\text{tautol.}}{=} L(\rho,s) \underset{\text{Euler prod for}}{=} \zeta(s) \underset{\text{Riemann's big discovery}}{=} L(f,s)$

$\zeta(s) = \sum n^{-s} = L(\mathbb{1},s)$

$\rho = \text{triv}$

$f = \theta(z) = \sum_{n\geq 1} q^{n^2}, q = e$

$\chi = \mathbb{1}$

(EG2) (Quadratic reciprocity in a simple sit.)

$A = \mathbb{Z}[x,y]/(x^2+y^2-1)$ have $L(A,s) \underset{\text{triv comp}}{=} L(\rho,s) = L(\chi,s) \underset{\text{Hecke}}{=} L(f,s)$

$1 - \frac{1}{3^s} + \frac{1}{5^s} - \cdots = L(\chi,s)$

f: a sort of θ fcn

$\rho : G(\bar{\mathbb{Q}}/\mathbb{Q}) \to G(\mathbb{Q}(i)/\mathbb{Q}) \hookrightarrow \mathbb{C}^\times$, $G(\mathbb{Q}(i)/\mathbb{Q}) \cong \{\pm 1\}$

$\rho(\sigma_p) = \left(\frac{-1}{p}\right)$

$\mathbb{1} = \chi \to$

f = a sort of θ

$L(\rho,s) = L(\chi,s)$ by: Euler's prod + quadr rec

$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \chi(p)$

$\chi : \mathbb{Z} \setminus 2\mathbb{Z} \to \left(\frac{\mathbb{Z}}{4\mathbb{Z}}\right)^\times = \{\bar{1},\bar{3}\} \xrightarrow{\sim} \{-1,1\}$

Dir char

(R) [ ] is a "reciprocity step"

(EG3) Quadr rec & more generally "Artin rec"

(EG4) (G) "$\subset$" given by Hasse–Tate mods $\varprojlim E[\ell^n]$ for $A = \frac{\mathbb{Z}[x,y]}{(y^2 - \varphi_3(x))}$

or e't coh for arb A

(G) "$\supset$" needs hypotheses idea: varieties can be broken up in coh. pieces.

(EG5) (L) "$\supset$" Eichl-Sh

(L) "$\subset$" Wiles-Taylor) for

(*) in partic motivic L-fcns have fcnal equs (b/c auto L-fcns do)