The linear algebra of quantum mechanics

Axiom: $\mathcal{A}$ is a $\mathbb{C}$-alg

- $\mathcal{H}$ = set of states ψ of a physical system S; superposition axiom: $\mathcal{H}$ is a $\mathbb{C}$-lin space.
- $\mathcal{A}$ = set of measurable quantities (observables); eg: positions, momenta, $\underbrace{\text{etc.}}_{\text{energy}}$
- For each $A \in \mathcal{A}$ one has a set $\sigma(A) \subset \mathbb{R}$, the set of all possible values of A
- For each $A \in \mathcal{A}$ one has a set $\pi(A) \subset \mathcal{H}$, the set of pure states for A (*)
- If S is in $\underbrace{\text{state}}_{\text{a pure}}$ ψ_n^A & a measurement of A is performed then the value of measurement is λ_n^A & S moves to a state equiv to ψ_n^A (i.e $\in \mathbb{C}\cdot\psi_n^A$)
- If S is in a state $\psi = \sum a_n \psi_n^A$ (w/ $a_n \in \mathbb{C}$, ψ_n pure) & a meas of A is performed then the value of meas is one of the λ_n^As & S moves to a state eq to ψ_n. The $\underbrace{\text{probality}}_{\text{amplitude}}$ that this happens for an n is a_n provided ψ is normalized (i.e $\sum |a_n|^2 = 1$). The prob is $p_n = |a_n|^2$. The average value of A (for state ψ) is $\sum |a_n|^2 \lambda_n^A = \sum p_n \lambda_n^A$ Call $a_n = \langle \psi, \psi_n^A \rangle$.
- Clearly $\langle \, , \, \rangle$ linear in 1st arg; also $\langle \psi_m^A, \psi_n^A \rangle = 0$ for $m \neq n$ & $=1$ for $m=n$.
- Axiom: $\langle \, , \, \rangle$ Hermitian inner prod. (so $\{\psi_n^A\}_n$ orthonormal basis)
- Next aim: justify why "$\mathcal{A}$ acts on $\mathcal{H}$" is reasonable. Indeed: if I define $A(\underbrace{\sum a_n \psi_n^A}_{\psi}) = \sum a_n \lambda_n^A \psi_n^A$ then A is Hermitian (b/c $\lambda_m^A \in \mathbb{R}$) & $\langle A\psi, \psi \rangle = \sum \lambda_n p_n$
- computing prob (amp) that system S goes from ψ_k^C to ψ_m^A after measuring first B & then A. It's $\sum_m \text{prob}_{\text{amp}}(\psi_k^C \to \psi_m^B)\, \text{prob}_{\text{amp}}(\psi_m^B \to \psi_n^A) = \sum_m \langle \psi_k^C, \psi_m^B \rangle \langle \psi_m^B, \psi_n^A \rangle$

So if $U_{CA} = (\langle \psi_n^C, \psi_m^A \rangle)$

then $U_{CA} = U_{CB} U_{BA}$

(*) A map $\pi(A) \to \sigma(A)$ is given. Axiom $\{\psi_m^A\}_{m \geq 1}$ gen $\mathcal{H}$ (topologically?) & lin ind.

$$\begin{array}{ccc} \psi_m^A & \mapsto & \lambda_m^A \end{array}$$