"DEDUCTION" OF LORENZ TRANSFORMATIONS (2 var's) & OF $E=mc^2$

Def $L := L(2) = \{A \in SL(2,\mathbb{R}) \mid q(Av) = q(v), \forall v \in \mathbb{R}^2\}$, $q(v) = v^t \begin{bmatrix}-1&0\\0&1\end{bmatrix} v$.

$\underset{\text{easy}}{=} \{A \in SL(2,\mathbb{R}) \mid \langle Av_1, Av_2\rangle = \langle v_1, v_2\rangle, \forall v_i\}$, $\langle v_1, v_2\rangle = v_1^t \begin{bmatrix}-1&0\\0&1\end{bmatrix} v_2$

$\underset{\text{easy}}{=} \left\{ \begin{bmatrix}\alpha&\beta\\ \beta&\alpha\end{bmatrix} \mid \alpha^2 - \beta^2 = 1 \right\}$

Notation $p = \binom{t}{x}$ Call its velocity (*) $vel(p) = \frac{x}{t}$ for $t \neq 0$.

Def For $A \in SL(2,\mathbb{R})$ say A has velocity $u \in \mathbb{R}$ if A sends $x = ut$ into $x' = 0$ where $\binom{t'}{x'} = A\binom{t}{x}$.♡ Write $vel(A) = u$. (*)
In other words if $vel(p) = u$ then $vel(A(p)) = 0$.

Def Let $\mathcal{L} \subset SL(2,\mathbb{R})$ be the set of all A s.t

1) $vel(A^{-1}) = -vel(A)$ ("physically" clear (*))
2) A sends $x = t$ into $x' = t'$ ("phisically" photons have velocity $=1$)

Prop 1 $\mathcal{L} = L$

Proof. Let $A = \begin{bmatrix}\alpha&\beta\\ \gamma&\delta\end{bmatrix} \in SL(2,\mathbb{R})$. Have $x' = \gamma t + \delta x$. ♡ So $x' = 0 \Leftrightarrow \gamma t + \delta x = 0$
So (if $\delta \neq 0$) $vel(A) = -\frac{\gamma}{\delta}$. Since $A^{-1} = \begin{bmatrix}\delta&-\beta\\ -\gamma&\alpha\end{bmatrix}$, $vel(A^{-1}) = \frac{\gamma}{\alpha}$. So cond 1) is $\Leftrightarrow$ to $\boxed{\alpha = \delta}$. Cond 2) is $\Leftrightarrow$ to "line $x' = t'$, which is $\alpha t + \beta x = \gamma t + \delta x$, equals line $x = t$" hence is $\Leftrightarrow$ to $\boxed{\beta - \delta = \gamma - \alpha}$.
So 1)+2) $\Leftrightarrow$ $\boxed{\alpha = \delta \ \& \ \beta = \gamma}$ □.

Rem 1 If $A = \begin{bmatrix}\alpha&\beta\\ \beta&\alpha\end{bmatrix}$, $\underset{= -\frac{\beta}{\alpha}}{vel(A)} = u$, then $\beta = -u\alpha$ so $A = \begin{bmatrix}\alpha&-u\alpha\\ -u\alpha&\alpha\end{bmatrix}$
where $\alpha^2 - u^2\alpha^2 = 1$ so $\alpha^2 = \frac{1}{1-u^2}$ so $\alpha = \frac{1}{\sqrt{1-u^2}} =: \alpha_u$ so $A = \underset{A_u}{\underset{\shortparallel}{\begin{bmatrix}\alpha_u&-u\alpha_u\\ -u\alpha_u&\alpha_u\end{bmatrix}}}$

Rem 2 $\alpha_{\frac{v-u}{1-vu}} = \alpha_u \alpha_v (1-uv)$ [trivial to check!]

Prop 2 $A_u \begin{pmatrix}\alpha_v\\ \alpha_v v\end{pmatrix} = \begin{pmatrix}\alpha_{\frac{v-u}{1-vu}}\\ \alpha_{\frac{v-u}{1-vu}} \cdot \frac{v-u}{1-vu}\end{pmatrix}$ [Triv to check using Rem 2]

♡ $t' = \alpha t + \beta x$
$x' = \gamma t + \delta x$

A "physically" interpr as a change of coord from frame F to frame F'

(*) $x' = 0$ "physically" interpreted as particle @ rest in F'
$x = ut$ — " — — " — moving w/ vel u in F.

Rem 3 If $\mathrm{vel}(P) = v$ then $\mathrm{vel}(A_u(P)) = \frac{v-u}{1-vu}$ (*)

Pf. $P = \begin{pmatrix} t \\ x \end{pmatrix} \Rightarrow A_u(P) = \alpha_u \begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = \alpha_u \begin{pmatrix} t-ux \\ -tu+x \end{pmatrix} \Rightarrow \mathrm{vel}(A_u(P)) = \frac{-tu+x}{t-ux} = \frac{-u+v}{1-uv}$

(**) **Def** An energy-momentum theory is a pair of fcns (φ,ψ), $\varphi,\psi : \mathbb{R}\to\mathbb{R}$ s.t. $\forall m_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, $w_i \in \mathbb{R}$, $u \in \mathbb{R}$ have

$$\mathrm{I}: \begin{cases} \sum m_i \varphi(v_i) v_i \overset{(1)}{=} \sum m_i \varphi(w_i) w_i \\ \sum m_i \psi(v_i) \overset{(2)}{=} \sum m_i \psi(w_i) \end{cases} \overset{(\heartsuit)}{\Longrightarrow} \mathrm{II}: \begin{cases} \sum m_i \varphi\left(\frac{v_i-u}{1-v_iu}\right)\frac{v_i-u}{1-v_iu} = \sum m_i \varphi\left(\frac{w_i-u}{1-w_iu}\right)\frac{w_i-u}{1-w_iu} \\ \sum m_i \psi\left(\frac{v_i-u}{1-v_iu}\right) = \sum m_i \psi\left(\frac{w_i-u}{1-w_iu}\right) \end{cases}$$

Prop 3 The pair (φ,ψ), $\varphi(v) = \psi(v) = \alpha_v = (1-v^2)^{-1/2}$ is an energy-mom theory.

Pf. I says $\sum m_i \begin{pmatrix} \alpha_{v_i} \\ \alpha_{v_i} v_i \end{pmatrix} = \sum m_i \begin{pmatrix} \alpha_{w_i} \\ \alpha_{w_i} w_i \end{pmatrix}$. Apply A_u. Get II by Prop 2.

Problem Find conditions which make $\varphi = \psi = \alpha_v$ the only solution (up to a mult scalar). (see Rem 6)

Rem 4 If φ, ψ as in Prop 3 one interprets $\underbrace{m\varphi(v)}_{m_v}$ as the relativistic mass (in order to have $p_v = m_v \cdot v$). Then $\boxed{E_v = m_v}$. This is the equiv of mass & energy.

Rem 5 Let $v = v(t)$. Then $\frac{d}{dt}(E_{v(t)}) = \frac{d}{dt}(p_{v(t)}) \cdot v(t)$. (o) which is ONE MORE justification for viewing E_v as energy & p_v as impulse.

Pf. $\frac{d}{dt}\left((1-v(t)^2)^{-1/2}\right) = \left[\frac{d}{dt}\left((1-v(t)^2)^{-1/2} v(t)\right)\right] \cdot v(t)$ [trivial to check]

Rem 6 **Def** A strong en-mom th is a pair (φ,ψ) s.t. $A_u\begin{pmatrix} \psi(v) \\ \varphi(v)v \end{pmatrix} = \begin{pmatrix} \psi\left(\frac{v-u}{1-vu}\right) \\ \varphi\left(\frac{v-u}{1-vu}\right)\frac{v-u}{1-vu} \end{pmatrix}$ $\forall u,v$. Note $\forall$ strong th is a th.

Prop $\exists!$ strong (φ,ψ) w/ $\psi(0) = 1$

Pf. Take $u^2 = 0$. Then $\alpha_u = 1$ ($\diamond$) so $\begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix}\begin{pmatrix} \psi(v) \\ \tilde\varphi(v) \end{pmatrix} = \begin{pmatrix} \psi(v) - u\psi'(v)(1-v^2) \\ \tilde\varphi(v) - u\tilde\varphi'(v)(1-v^2) \end{pmatrix} \Rightarrow$

Set $\tilde\varphi(v) = \varphi(v)\cdot v$

$$\begin{cases} \psi'(v)(1-v^2) = \tilde\varphi(v) \\ \tilde\varphi'(v)(1-v^2) = \psi(v) \end{cases} \Rightarrow \text{eqn of ord 2 in } \psi \text{ w/ given } (\psi(0), \psi'(0) = \tilde\varphi(0)) \Rightarrow \psi \text{ unique.}$$

$\tilde\varphi'(0) = \varphi(0) = 1$, $\tilde\varphi(0) = 0$

Rem Easy to generalize to 4 variables $p_{\vec v} = m\varphi(v)\cdot\vec v$, $E_v = m\psi(v)$; $\frac{d}{dt}(E_v) = \frac{d}{dt}(p_{\vec v})\cdot\vec v$.

(o) as in the classical case $\frac{d}{dt}\left(\frac{m v(t)^2}{2}\right) = \frac{d}{dt}(m v(t))\cdot v(t)$.

(**) Equality (1) says "conservation of momentum" & (2) "cons of energy" where $\overbrace{m\varphi(v)v}^{p_v}$ is the mom of a particle of "rest mass" m & $\underbrace{m\psi(v)}_{=:E_v}$ is the energy. ($\heartsuit$) says that if conserv holds in one frame it holds in any other.

(*) If $P' = A_u(P)$ the rem gives rel b/t velocities in 2 frames of rel vel u.

($\diamond$) $u^2 = 0 \Rightarrow \frac{v-u}{1-vu} = v - u(1-v^2)$.