CORRESPONDENCES ON CURVES, FERMAT QUOTIENTS, AND UNIFORMIZATION

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Abstract. The quotient of a curve by a correspondence usually reduces to a point in algebraic geometry. One way to fix this pathology is to extend algebraic geometry in a "non-commutative" direction. Another way (which is the subject of this talk) is to extend algebraic geometry by staying within the commutative setting but adjoining instead a new operation: the Fermat quotient. It turns out that in this new geometry a number of interesting quotients of curves by correspondences become non-trivial and indeed rather interesting. The examples that can be treated in this way arise from correspondences that can be "analytically uniformized". Remarkably these are closely related to some of the main examples treated via non-commutative geometry.

1. Motivation

Cat: $\mathcal{C}$
Corr: (X,σ), $\sigma=(\sigma_1,\sigma_2)$, $\sigma_1,\sigma_2:\tilde{X}\to X$.
Cat Quot: $(X/\sigma,\pi)$, $\pi:X\to X/\sigma$, $\pi\circ\sigma_1=\pi\circ\sigma_2$, universal.
[Basic pathology] BP: $\mathcal{C}=\{\text{algebraic varieties}\}/\{\text{smooth manifolds}\}$;
σ with a dense orbit $\Rightarrow X/\sigma=pt$ ("trivial")
[What to do? Nothing OR search for new geometries where BP may disappear]

2. Two strategies

strategies	principle	BP persists	BP treatable
invariant th (eff descent)	fcns on X/σ are fcns φ on X s.t. $\varphi\circ\sigma_1=\varphi\circ\sigma_2$	$\mathcal{C}=\{\text{alg var}\}$	$\mathcal{C}_\delta=\{\delta\text{-geo}\}$ (A.B.)
groupoid th (non-eff descent)	individual fcns on X/σ not defined modules on X/σ are modules on X plus descent data	$\mathcal{C}_{st}=\{\text{alg stacks}\}$	$\mathcal{C}_{nc}=\{\text{nc-geo}\}$ (Connes)

$\mathcal{C}_{st}$: descent data on modules encoded into data rel to diagram of comm rings
$\mathcal{C}_{nc}$: descent data on modules encoded into

structure of module over nc (convolution) ring
$\mathcal{C}_\delta$: gen by $\mathcal{C}$ and ONE new mor δ=Fermat quotient, all rings comm

3. Plan

Say $\mathcal{C} = \{\text{alg varieties}\}$
introduce $\mathcal{C}_\delta$
attach to corr (X, σ) in $\mathcal{C}$ a corr $(X_\delta, \sigma_\delta)$ in $\mathcal{C}_\delta$
show classes of EGs with X/σ trivial in $\mathcal{C}$ and X_δ/σ_δ non-trivial in $\mathcal{C}_\delta$
study geo and coho of X_δ/σ_δ
Remark: $\delta-$geo and non-commutative geo
succeed in related EGs!!! Coincidence?
Will proceed in 2 rounds: 1st round vague, 2nd round more precise

4. δ-geo, 1st round

$R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N, p) = 1]\hat{}$
[Superscript $\hat{}$ means p-adic completion]
$\phi : R \to R$ [unique ring homo with] $\phi(x) \equiv x^p \mod p$
$\delta : R \to R$, $\delta x = \frac{\phi(x)-x^p}{p}$ Fermat quotient [operator] $\delta =$ "$\frac{d}{dp}$"
"constants" $\{x \in R; \delta x = 0\}$ are zero and roots of 1
LOOKS LIKE A GEOMETRY OVER $\mathbb{F}_1$!!!!!!!!
X affine smooth over R, $X \subset \mathbb{A}^n$
$f : X(R) \to R$ δ-function of order r if
$f(x) = F(x, \delta x, ..., \delta^r x), \quad x \in X(R) \subset R^n$,
$F \in R[T, T', ..., T^{(r)}]\hat{}$
$\mathcal{O}^r(X)$ ring of δ-functions
δ-geometry: a geometry with
objects (morally, locally, intersections of) $f^{-1}(0)$,
morphisms: $(g/h)|f^{-1}(0)$
$f, g, h \in \mathcal{O}^r(X)$

5. Main results, 1st round

	δ-geometry	non-commutative geometry
spherical	$\frac{\mathbb{P}^1(R)}{SL_2(\mathbb{Z}_p)}$	$\frac{\mathbb{P}^1(\mathbb{R})}{SL_2(\mathbb{Z})}$ = non-comm mod curve
flat	$\frac{E(R)}{\langle\gamma_i\rangle}, \frac{E(R)}{[n]}$	$\frac{S^1}{\langle e^{2\pi i\theta}\rangle}$ = non-comm ell curve
hyperbolic	$\Gamma\backslash\mathbb{H} = Sh_\Gamma \to \frac{Sh_\Gamma}{\text{Hecke}}$	$\lim Sh_\Gamma = Sh^0 \subset Sh \subset Sh^{(nc)}$

6. δ-geo: 2nd round

[Usual path in geometry: ringed spaces; with δ-geometry: difficulties
Two choices: go more sophisticated or more naive; we choose to go naive]
R ring
Ringed set $X_* = (X_{set}, S = X_{mon}, (X_s)_{s\in S}, (\mathcal{O}_s)_{s\in S})$
X_{set} set, X_{mon} monoid,
$X_s \subset X_{set}$, $X_{st} = X_s \cap X_t$,
$\mathcal{O}_s \subset \{\text{maps } X_s \to R\}$ subrings,
$f \in \mathcal{O}_s \Rightarrow f|X_{st} \in \mathcal{O}_{st}$
X_* trivial if $\mathcal{O}_s = R$, all s.
[Ring of rational functions] $R\langle X_*\rangle := \lim_{\to} \mathcal{O}_s$
[Morphisms] $f_* : X_* \to Y_*$ of ringed sets: $f_* = (f_{set}, f_{mon})$ s.t....
$\{$ringed sets$\}$ has categorical quotients:
$X_*/\sigma_* = (X_{set}/\sigma_{set}, X_{mon}/\sigma_{mon}, ...)$
[NOW:] $\delta : R \to R$ map of sets
X_* called δ-ringed set if
$f \in \mathcal{O}_s \Rightarrow \delta \circ f \in \mathcal{O}_s$
$\delta : R\langle X_*\rangle \to R\langle X_*\rangle$
Full [subcat] $\{\delta$-ringed sets$\}$; has categorical quotients
Examples
k a.c. field
R is k or $W(k)$
X/R variety resp smooth scheme [with irr geo fibers]
$X = \cup_{i\in I} X_i$ affine [cover]
$X_! = (X(R), \mathcal{P}_0(I), (X_J(R)), (\mathcal{O}(X_J)))$
[$\mathcal{P}_0(I)$ finite subsets of I, monoid under union, index J means $\cap_{j\in J}$]
[ringed set but not a δ-ringed set in general unless δ poly map, say]
From now on $R = W(\overline{\mathbb{F}}_p)$, $\delta : R \to R$, $\delta x = \frac{\phi(x) - x^p}{p}$
$w = \sum a_i \phi^i \in W = \mathbb{Z}[\phi]$ acts on $R^\times$ by $\lambda^w = \prod \phi^i(\lambda)^{a_i}$.

7. Functor from alg geometry to δ-geometry

[Will define functor]
$\{$smooth R-schemes + etale maps$\} \to \{\delta$-ringed sets$\}$
[assume irr geo fibers]
$X \mapsto X_\delta = (X(R), ...)$
affine cover $X = \cup_{i\in I} X_i$,
choose cocycle $\lambda_{ij} \in \mathcal{O}(X_{ij})^\times$ (hence line bundle L)
For w of "order" r
$A_w := \{(f_i); f_i \in \mathcal{O}^r(X_i), f_i = \lambda_{ij}^w f_j\}$ [a space of "sections" of a "bundle"]
Ring $A = \bigoplus_{w\in W} A_w$, $W_+ =$ the $w \in W$ with coeff ≥ 0
$S = X_{mon} = \cup_{0\neq w\in W_+}(A_w \backslash pA_w)$
$X_{set} = X(R)$
For $s = (f_i) \in A_{w_0}$
$X_s = \bigcup_{i\in I}\{P \in X_i(R); f_i(P) \not\equiv 0 \mod p\}$
$\mathcal{O}_s = \{P \mapsto \frac{g_i(P)}{f_i^w(P)}; (g_i) \in A_{ww_0}\}$
X_δ [defined by data above, a sort of "Proj A"] depends on L
[To make this functorial in etale maps] can take $L = K^{-1}, \mathcal{O}, K$

Will take $L = K^{-1}$ [the other 2 choices bad for what's next]
N.B. X_δ depends only on $X_!$, not on X.

8. Main conjectures

(X, σ) corr in {alg varieties over $\overline{\mathbb{Q}}$}
$X, \tilde{X}$ affine curves, σ etale
View as corr over R for $p >> 0$
$(X, \sigma) \mapsto (X_\delta, \sigma_\delta) \mapsto X_\delta/\sigma_\delta = (X(R)/\sigma(R), ...)$
Direct Conjecture:
if (X, σ) has an analytic uniformization (cf. below)
then after deleting fin many points from (X, σ)
X_δ/σ_δ is non-trivial and "δ-rational" for $p >> 0$.
Latter $\Rightarrow (R\langle X_\delta/\sigma_\delta\rangle)\hat{} \simeq (R\langle \mathbb{A}^1_\delta\rangle)\hat{}$.
Reasonable: X_δ/σ_δ "δ-Fano" b/c of K^{-1}.
Converse Conjecture:
if X_δ/σ_δ is non-trivial for $p >> 0$
then (X, σ) has an analytic uniformization.

9. Analytic unifromization of correspondences

(X, σ) has analytic uniformization if, after adding fin many points:

$$\begin{array}{ccccc} \Sigma & \leftarrow & \Sigma & \rightarrow & \Sigma \\ \downarrow & & \downarrow & & \downarrow \\ X & \leftarrow & \tilde{X} & \rightarrow & X \end{array}$$

Σ simply connected Riemann Surf: sphere, flat plane, hyp plane
top arrows iso,
vertical arrows ramified Galois with group: finite, infinite, finite covol
Can be classified:
spherical $\Rightarrow$ platonic
flat $\Rightarrow$ multiplicative, Chebyshev, Lattes
hyperbolic $\Rightarrow$ vertical groups arithmetic (Margulis)
so $X, \tilde{X}$ Shimura curves
attached to quaternion algebras over totally real fields F

10. Main results, 2nd round

Direct Conjecture true for 1) spherical, 2) flat, 3) hyp ($F = \mathbb{Q}$, Hecke corrs)
Moreover: [detailed study of] geometry and cohomology of X_δ/σ_δ
[Math behind: study of USUAL alg geo of certain formal schemes
("arith jet spaces") attached to ell curves, modular forms, etc]
Converse Conjecture true for X rational and $\sigma_1 = id$
[$\tilde{X}$ graph of an endomorphism]
[Math behind: classification of self maps of $\mathbb{P}^1$ with
invariant tensorial forms]