Schemes | Moduli schemes

(D) For A a ring$^{(*)}$, an A-fctor is a fctor $X : \{A\text{-alg}\} \to \{\text{sets}\}$
an A-group is a fctor $X : \{A\text{-alg}\} \to \{\text{groups}\}$

(D) Mor of A-fctors, of A-grs are mor of fctors.

(EG1) For $\underbrace{F_1, \dots, F_m}_{F} \in A[x_1, \dots, x_n]$, $X = X_F$, $X(B) = \{\underbrace{(b_1, \dots, b_n)}_{b} \in B^n / F_j(b) = 0, \text{all } j\}$

(EG2) For A-alg S, $X = \operatorname{Spec} S$, $X(B) = \operatorname{Hom}_{A\text{-alg}}(S, B)$ ♡ Note ↑ special case when S f-pres / A.

(D) An aff sch$/_A$ is an A-fctor iso to a Spec S. An aff sch/A of fin type is .... fin/A is ....

(EG3) For A-alg S & ideal $I \subset A$ have $V(I) \overset{\text{def}}{=} \operatorname{Spec} \frac{A}{I} \to \operatorname{Spec} A$; any sit iso to this called: closed subsch of aff sch.

(EG4) For A & S & I as ↑ define $X(B) = \{P : S \to B \text{ of } A\text{-alg}; \ P(I) \cdot B = B\}$; call $X = D(I) \overset{\text{:=}}{=} \operatorname{Spec} A \setminus V(I)$; any sit iso to this called open subsch or q-affin etc.

(EG5) For $F_1, \dots, F_m \in A[x_0, \dots, x_n]$ homog of deg d_j, $X = X_F$, $X(B) = \{(m_0, \dots, m_n) \overset{=m}{\in} M^{n+1}$; M fin gen proj B-mod of rk 1, $\sum B m_i = M$, $F_j(m) = 0 \in M^{\otimes d_j}\} / \text{iso}$

(D) A proj sch is an X iso to one as ↑. (D) $\not\exists V(I)$ & cl. imm.

(EG6) For grad fin gen A-alg S w A Noeth, $X = \operatorname{Proj} S$ defined via ↑. Def $S^{(d)} \overset{\text{def}}{=} \bigoplus_i S_{di}$

Note $\operatorname{Proj} S = \operatorname{Proj} S^{(d)}$.

If $S = S_0[S_1]$ $\left(= \bigoplus_{i \geq 0} S_i = S_0[S_1]\right)$ dropped def $\operatorname{Proj} S = \operatorname{Proj} S^{(d)}$ for d big.

(EG7) If $I \subset S$ graded id, $X = D(I)$ $(:= \operatorname{Proj} S \setminus V(I))$, $X(B) = \{(m_0 : \dots : m_n) \in \operatorname{Proj} S \mid \sum_{G \in I^h} G(m) B = M^{\otimes \deg G}\}$

Any sit iso to this called open imm, q-proj etc.

(EG8) $\mathbb{G}_a, \mathbb{G}_m, E_{a,b} = \operatorname{Proj} A[x_0, x_1, x_2] / (x_0 x_2^2 - x_1^3 - a x_1 x_0^2 - b x_0^3)$, $(a, b) \in A^2$, $\Delta = 4a^3 + 27b^2 \in A^\times$

(they are A-grs) ↑ called plane ell curve.

(P) $E_{a,b} \simeq E_{a',b'}$ (as proj schs) $\iff \exists \lambda \in A^\times$ s.t. $a' = \lambda^4 a$, $b' = \lambda^6 b$. (Call i_λ the iso of ell curves)

(EG9) $\mu_{N,A} = \operatorname{Spec} A[x] / (x^N - 1)$, $(\mathbb{Z}/N\mathbb{Z})_A(B) = \mathbb{Z}/N\mathbb{Z}$ (aff sch!)

$\mu_{N,A}(B) = \{b \in B \mid b^N = 1\}$. Note $\mu_{N,A} \simeq (\mathbb{Z}/N\mathbb{Z})_A$ if A field of ch $\nmid N$. But $\not\simeq$ if ch $\mid N$.

(D) Fiber prods of fctors; kernels of A-gr homos defined as ____ fibprods.

(EG) Fiber prod of aff sch is aff sch (given by $\otimes$); same for projective.

(EG) $E[N] = \operatorname{Ker}([N] : E \to E)$ for $E = E_{a,b}$; (P) $E[N]$ both proj & aff fin/A. So kers are closed subsch.

(EG) $\mathcal{M}_{\Gamma_1(1)} = \operatorname{Spec} M_{\Gamma_1(N)}$, $M_{\Gamma_1(N)} = A[a_4, a_6, \frac{1}{\Delta}]$ w/ a_4, a_6 letters. Then $\mathcal{M}_{\Gamma_1(1)}(B) = \{(\alpha, \beta) \in B^2 / 4\alpha^3 + 27 b^2 \in B^\times\}$ (here $\mathbb{Z}[\frac{1}{6}] \to A$)

(*) rings are comm w/ 1.

♡ discuss representable fctors. $\forall$ 2 obj repres a fctor are iso

$\mathbb{Z}[\frac{1}{6}]$ $\mathcal{R}_{\Gamma_1(1)}(B)$ cl. imm of aff B-schemes

(EG) $\mathcal{R}_{\Gamma_1(N)}$ A-fctor def by $\mathcal{R}_{\Gamma_1(N)}(B) = \{(\alpha,\beta), \left(\frac{\mathbb{Z}}{N\mathbb{Z}}\right)_B \overset{i}{\hookrightarrow} E[N]_{\alpha,\beta} /_B \}$

(T) $\mathcal{R}_{\Gamma_1(N)}$ is an aff scheme of fin type / $A = \mathbb{Z}[\frac{1}{6}]$ & fin / $\mathcal{R}_{\Gamma_1(1)}$.

Pf. Let $\mathbb{E} = E_{a_4 a_6}$ (w/ $a_4 a_6$ letters). Then one checks.

$$\mathcal{R}_{\Gamma_1(N)}(B) \cong \left(\mathbb{E}[N] \setminus \bigcup_{d \mid N} \mathbb{E}[d]\right)(B)$$

(where $\mathbb{O}\setminus\mathbb{O}$ defined as $D(I)$ ... so q-aff but one act. pves affine)

(D) Action of A-gr on an A-fctor

(P) If $\mathbb{G}_m$ acts on $X = \operatorname{Spec} S$ then $S = \bigoplus_{d \in \mathbb{Z}} S_d$, $S_d = \{a \in S \mid a \mapsto at^d \text{ via } S \to S[t]\}$.

(EG) $\mathbb{G}_m$ acts on $\mathcal{R}_{\Gamma_1(N)}$ $\left(\mathbb{G}_m(B) \circlearrowright \mathcal{R}_{\Gamma_1(N)}(B),\ (\alpha,\beta;i) \mapsto (\lambda^4\alpha, \lambda^6\beta; \underbrace{\mathrm{iso}_\lambda \circ i}_{i_\lambda})\right)$

so $M_{Y_1(N)} = \bigoplus_{k \in \mathbb{Z}} M_{Y_1(N)}(k) = \bigoplus_{k \in \mathbb{Z}} M_{Y_1(N)}(k, A)$

(R) An elt $f \in M_{Y_1(N)}(k)$ defines & is defined by a rule $\mathcal{R}_{\Gamma_1(N)}(B) \to B = \mathbb{A}^1(B)$

s.t $f(\lambda^4\alpha, \lambda^6\beta; i_\lambda) = \lambda^{-k} f(\alpha,\beta;i)$ (b/c $f \in M_{Y_1(N)}$ a mor $\mathcal{R}_{\Gamma_1(N)} \to \mathbb{A}^1$) analog of deg k

(D) $Y_1(N) = \operatorname{Spec} M_{Y_1(N)}(0)$

(P) (Shimura, Igusa) $\underbrace{Y_1(N)}_{\text{for } N \geq 4}$ represents the fctor $B \mapsto \{(\alpha,\beta;i)\}/\text{iso}$

(For $A = \mathbb{Z}[\frac{1}{6}]$)

(D) Let $M_{X_1(N)}$ be the int clos of $A[a_4 a_6]$ in $M_{Y_1(N)}$. Then

$M_{X_1(N)}$ is graded (easy). Def $X_1(N) = \operatorname{Proj}(M_{X_1(N)})$.

(P) $X_1(N)(\mathbb{C}) = \Gamma_1(N) \setminus \mathbb{H}^*$

(D) For R any $A = \mathbb{Z}[\frac{1}{6}]$-alg let $M_{X_1(N)/R} = M_{X_1(N)} \otimes_A R$

& $X_1(N)_{/R} := \operatorname{Proj}(M_{X_1(N)/R})$.

(*) $\mathcal{R}_{\Gamma_1(N)} = \operatorname{Spec} M_{Y_1(N)}$

(σ) $M_{X_1(N)} = \bigoplus_{k=0}^{\infty} \overbrace{M_{X_1(N)}(k, A)}^{\text{called mod fns of wt } k \text{ on } \Gamma_1(N) \text{ over } A}$