Plane curves

$k = \bar{k}$ field, $\mathbb{P}^2 = \mathbb{P}(k^3)$

S_m = { polynomials in $k[x_0, x_1, x_2]$, homog. of deg m }

$|mL| := \mathbb{P}(S_m)$ $^{(\nabla)}$; define for $P \in \mathbb{P}^2$, $C \in |mL|$ incidence $P \in C$ $^{(+)}$

Def For $C \in |mL|$, $D \in |nL|$, $P \in C \cap D$ define $(C \cdot D)_P = \dim_k \cdots$

Th (Bézout) C, D as above; then $\sum_{P \in C \cap D} (C \cdot D)_P = mn$ provided C & D have no "component in common" (i.e. no common factor).

Def For $P_1, \dots, P_r \in \mathbb{P}^2$ define $|mL - P_1 - \dots - P_r|$

(L) $\dim |mL - P_1 - \dots - P_r| = \dim |mL - P_1 - \dots - P_{r-1}| - \delta$, $\delta \in \{0, 1\}$

Prop Let $P_1, P_2, P_3, P_4, P_5 \in \mathbb{P}^2$ be s.t. no 4 collinear. Then $\exists!$ quadric $Q \in |2L|$ passing thr. them.

Pf. If $\exists$ 3 collinear, say P_1, P_2, P_3 take $Q = L_{13} + L_{45}$ $^{(**)}$

Assume no 3 collinear. By (L) need $|2L - P_1 - \dots - P_r| \neq |2L - P_1 - \dots - P_{r-1}|$ for $r = 1, \dots, 5$. Hardest case $r = 5$. Need $Q \in |2L|$ passing thr $P_1, \dots, P_4$ but not P_5. Take $Q = L_{12} + L_{34}$

Thm Let $C, D \in |3L|$ meet in $P_1, \dots, P_9$. Let $E \in |3L|$ pass thr. $P_1, \dots, P_8$. Then $P_9 \in E$.

Remark. If 4 of $P_1, \dots, P_9$ on a line L then Bézout $\Rightarrow$ L comp of C & D $\lightning$. So no 4 pts collinear. (Also claim no 7 pts of the 9 on a conic Q. Indeed if $Q = L_1 + L_2 \Rightarrow \exists$ at least 4 on an L_i $\lightning$. If Q irred, by Bézout Q comp of C & D $\lightning$.)

(∇) $\dim |2L| = 5$, $\dim |3L| = 9$

$(**)$ L_{ij} = line thr. P_i & P_j. Unicity $\Leftarrow$ Bézout.

$(+)$ $P \in C \cap D$, etc. Also define $|mL| \times |nL| \to |(m+n)L|$, $(C, D) \mapsto C + D$.

<u>Pf. of Thm</u> Enough to show $\dim|3L - P_1 - \cdots - P_8| = 1$ (For then if $C = [F]$, $D = [G]$, $G, F, H \in S_3$ then $H \in \mathrm{Span}\{F, G\}$ & $\square$)
$E = [H]$

Enough to show each $|3L - P_1 - \cdots - P_r| \neq |3L - P_1 - \cdots - P_{r-1}|$ if $r \leq 8$

Harder case $r = 8$. Need cubic through $P_1, \ldots, P_7$ not passing thr. P_8.

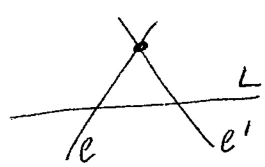
Let $C_i = Q_{1234i} + L_{jk}$, $\{i, j, k\} = \{5, 6, 7\}$. ($Q_{1234i}$ = unique quadr thr $L_1, \ldots, L_i$). Assume C_5, C_6, C_7 all pass thr P_8 & derive a $\lightning$.

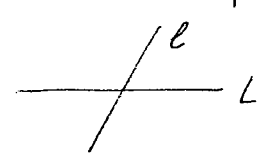
Note P_8 cannot lie on 2 of the 3 lines L_{67}, L_{57}, L_{56} (b/c we would get 4 collinear pts ~~among~~ P_5, P_6, P_7, P_8 $\lightning$)

So may assume $P_8 \notin L_{57}$, $P_8 \notin L_{67}$ hence

$$P_8 \in Q_{12346}, \quad P_8 \in Q_{12345}$$

~~Let $Q \neq Q_{12346}$~~ So the latter 2 quadrics meet in 5 pts. So they have a line L in common, so their $\cap$ is either $L \cup \{\text{point}\}$ or $L \cup \{\text{line}\}$

(
$Q_{12346} = L + \ell$
$Q_{12345} = L + \ell'$

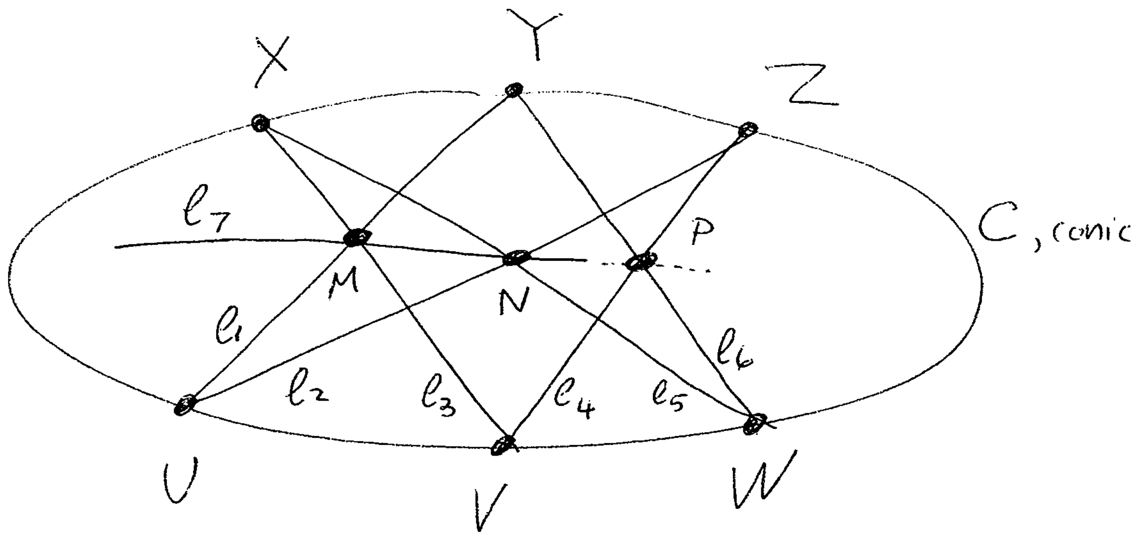
or 
$Q_{12346} = Q_{12345} = \ell_1$,

But $P_1, P_2, P_3, P_4, P_8 \in L \cup \{pt\} \Rightarrow$ 4 of the pts colinear $\lightning$
~~or lines~~

So must have $Q_{12346} = Q_{12345} = \ell + L$. But $P_1, P_2, P_3, P_4, P_5, P_6, P_8 \in \ell + L$

$\Rightarrow$ 4 of the pts on a line $\lightning$.

Pascal's Theorem (Pappus)



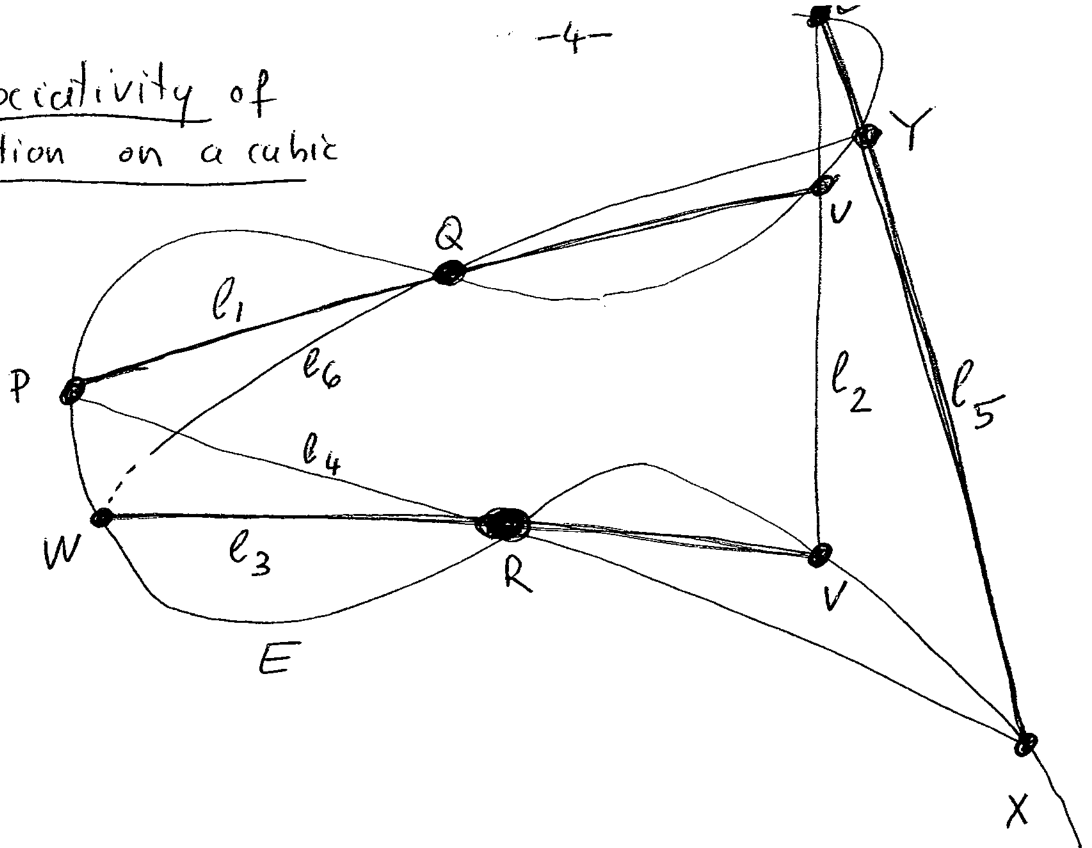
$\ell_4 + \ell_1 + \ell_5$ & $\ell_6 + \ell_3 + \ell_2$

pass through $P, V, Z, Y, M, U, W, X, N$

$C + \ell_7$ passes through all these but P

so it passes through $\underline{P}$

Associativity of addition on a cubic



$V = P \oplus Q$

$W = \ominus(R \oplus V)$

$Y = P \oplus R$

cubics E & $\ell_1 + \ell_3 + \ell_5$ pass through $P, Q, U, W, R, V, \mathcal{O}, Y, X$;

cubic $\ell_2 + \ell_4 + \ell_6$ passes through $\mathcal{O}, U, V, P, R, X, Y, Q$

so must pass through W as well